Received: *October 17, 2001*, Accepted: *November 20, 2001*

Quarter BPS operators in $\mathcal{N} = 4$ SYM

Anton V. Ryzhov

Department of Physics and Astronomy, University of California
Los Angeles, LA, CA 90095-1547, USA
E-mail: ryzhovav@physics.ucla.edu

Abstract: Chiral primary operators annihilated by a quarter of the supercharges are constructed in the four dimensional $\mathcal{N} = 4$ Super-Yang-Mills theory with gauge group $\mathrm{SU}(N)$. These $\frac{1}{4}$-BPS operators share many non-renormalization properties with the previously studied $\frac{1}{2}$-BPS operators. However, they are much more involved, which renders their construction nontrivial in the fully interacting theory. In this paper we calculate $\mathcal{O}(g^2)$ two-point functions of local, polynomial, scalar composite operators within a given representation of the $\mathrm{SU}(4)$ R-symmetry group. By studying these two-point functions, we identify the eigenstates of the dilatation operator, which turn out to be complicated mixtures of single and multiple trace operators. Given the elaborate combinatorics of this problem, we concentrate on two special cases. First, we present explicit computations for $\frac{1}{4}$-BPS operators with scaling dimension $\Delta \leq 7$. In this case, the discussion applies to arbitrary N of the gauge group. Second, we carry out a leading plus subleading large N analysis for the particular class of operators built out of double and single trace operators only. The large N construction addresses $\frac{1}{4}$-BPS operators of general dimension.

Keywords: AdS-CFT Correspondance, Conformal Field Models in String Theory, Conformal and W Symmetry.

JHEP11(2001)046

Contents

1. Introduction

During the past years, there has been a renewed interest in the study of chiral operators in the $\mathcal{N}$=4 supersymmetric Yang-Mills theory in four dimensions. Forming short representations of the global $\mathrm{SU}(2,2|4)$ superconformal symmetry group, chiral operators have tightly constrained quantum numbers. In particular, the scaling dimension of a chiral operator is not renormalized.[1]

Chiral primary operators have been classified in [1, 2]. They can be $\frac{1}{2}$-BPS, $\frac{1}{4}$-BPS, and $\frac{1}{8}$-BPS. The $\frac{1}{2}$-BPS operators provide the simplest example of chiral primaries. These are scalar composite operators in the $[0,q,0]$ representations of the R-symmetry group $\mathrm{SU}(4) \sim \mathrm{SO}(6)$; their scaling dimension is $\Delta = q$, see [2]. $\frac{1}{2}$-BPS operators are annihilated by eight out of the sixteen Poincaré supercharges of the theory. Similarly, $\frac{1}{4}$-BPS primaries belong to $[p,q,p]$ representations of the R-symmetry group, are annihilated by four supercharges, and have protected scaling dimension of $\Delta = 2p+q$. Finally, $\frac{1}{8}$-BPS primaries live in $[p,q,p+2k]$ of $\mathrm{SU}(4)$, are killed by only two supercharges, and their $\Delta = 3k+2p+q$. Quantum numbers of the descendant operators are related to those of their primaries by the $\mathcal{N}$=4 superconformal algebra.

$\frac{1}{2}$-BPS operators have been much studied. Using the conjectured AdS/CFT correspondence [3], it was shown, that for gauge groups $\mathrm{SU}(N)$ with N large, two and three point functions of $\frac{1}{2}$-BPS chiral primaries are the same at weak and strong coupling [4].[2] It was then verified that these correlators get no $\mathcal{O}(g^2)$ corrections on the SYM side, for arbitrary N [6]. Chiral descendant operators share these non-renormalization properties with their parent primaries [6]. Order g^4 and instanton contributions to two and three point functions of $\frac{1}{2}$-BPS chiral primaries turn out to vanish as well [7]–[10]. Nonrenormalization of these correlators was further established on general grounds in [11, 12]. Besides $\mathrm{SU}(N)$ theories and single trace chiral primaries, multiple trace operators with the same $\mathrm{SU}(2,2|4)$ quantum numbers, as well as arbitrary gauge groups were considered [15]. In these cases, two and three point functions were also found to receive no $\mathcal{O}(g^2)$ corrections.

[1] The possibility that certain non-chiral operators may have vanishing anomalous dimension was raised in [9].

[2] Higher n-point functions also agree with supergravity predictions in the large N limit [5].

It is natural to ask whether other chiral operators, for example $\frac{1}{4}$-BPS primaries, have protected correlators. Here the situation is much less straightforward than for $[0,q,0]$ operators. In fact, except for the simplest operator found in [8], no other $\frac{1}{4}$-BPS chiral primaries were written down[3] in the fully interacting theory. The main difficulty is that unlike in the free theory, where a kinematical (group theoretical) treatment of [2] is sufficient, for nonzero coupling the problem of determining primary operators becomes a dynamical question.[4]

Apart from the double trace scalar composite operators in the $[p,q,p]$ of the R-symmetry (flavor) group SU(4) (the free theory chiral primaries from the classification of [2]), there are other single and multiple trace scalar composite operators with the same SU(4) quantum numbers and the same Born level scaling dimension. Unlike in the $\frac{1}{2}$-BPS case where this phenomenon occurs [15], scalar composites in the $[p,q,p]$ generally do not have a well defined scaling dimension. Therefore, one should first find their linear combinations which are eigenstates of the dilatation operator, which we call pure operators. To this end, we calculate two point functions of local, gauge invariant, polynomial, scalar composite operators in a given $[p,q,p]$ representation; diagonalize the dilatation operator within each representation of SU(4); and find that some of the pure operators receive no $\mathcal{O}(g^2)$ corrections to their scaling dimension or normalization. These operators have the right SU(4) quantum numbers and protected $\Delta = 2p + q$, and are the only candidates for being the $\frac{1}{4}$-BPS chiral primaries from the classification of [2].

Calculating the symmetry factors for Feynman diagrams is a formidable combinatorial problem for general representation $[p,q,p]$ of SU(4), and general N of the gauge group SU(N). So to keep the formulae manageable, in this paper we concentrate on two special cases. For low dimensional operators ($2p + q < 8$), we perform explicit computations for arbitrary N; in particular, we recover the simplest $\frac{1}{4}$-BPS operator studied previously in [8]. Alternatively, we give a leading plus subleading large N argument (valid for general $[p,q,p]$ representations) for a class of $\frac{1}{4}$-BPS chiral primaries, which are linear combinations of double- and single-trace scalar composite operators.

The paper is arranged as follows. First we review some aspects of $SU(2,2|4)$ group theory, and describe the scalar composite operators we will be dealing with. Then we set the stage for $\mathcal{O}(g^2)$ calculations of two-point functions, and outline the main ingredients of these calculations. After that, we explicitly compute the simplest sets of correlators. In the course of these computations, it turns out that only one type of Feynman diagrams contributes to the correlators at order g^2, and we provide a simple explanation of this fact.[5] We present the full calculation for these two point

[3] $\frac{1}{4}$-BPS operators have been studied indirectly through OPEs of $\frac{1}{2}$-BPS chiral primaries, see [8, 9, 13, 14].

[4] I would like to thank Sergio Ferrara for bringing this to my attention.

[5] The argument we give applies more generally. In particular, it provides an alternative interpre-

functions. For higher Δ, calculations were done using *Mathematica* and only the results are shown. Several new features come into play, and we describe them as we go along. Finally, we switch gears and do a large N analysis of $\frac{1}{4}$-BPS operators with arbitrary scaling dimension.

In this paper, we properly identify the $\frac{1}{4}$-BPS primaries in the fully interacting theory. Analysis of two point functions is the first step in a systematic study of nonrenormalization properties of chiral operators. Three-point correlators of $\frac{1}{4}$-BPS primaries will be studied in [24].

2. SU(2, 2|4) group theory

Four dimensional $\mathcal{N}$=4 superconformal Yang-Mills theory has been studied extensively for a long time, and we begin by reviewing some well known facts.

$\mathcal{N}$=4 SYM can be formulated in several (equivalent) ways; see appendix A for some of the descriptions. None of them shows all the features of the theory explicitly. For example, working with six scalars $\phi^I = \phi^I_a t^a$ (where $a = 1, \ldots, N^2 - 1$ runs over the gauge group SU(N), and $\phi^I_a(x)$ are real scalar fields), and grouping the fermions as λ^i_a, $i = 1, \ldots, 4$, makes the full SU(4) R-symmetry group manifest, but hides all the supersymmetries. On the other hand, formulating the theory in terms of $\mathcal{N}$=1 superfields shows some of the supersymmetry, but the lagrangian looks invariant just under the SU(3) $\times$ U(1) subgroup of the full SU(4). In practice, the more supersymmetries we use, the simpler it is to perform actual calculations. [6] For our purposes it suffices to use the component fields of the $\mathcal{N}$=1 superfield formulation of the theory, with the (euclidean signature) lagrangian [6]

$$\begin{aligned}\mathcal{L}=\mathrm{tr}\,&\left\{\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}\bar\lambda\gamma^\mu D_\mu\lambda + \overline{D_\mu z_j}D^\mu z_j + \frac{1}{2}\bar\psi^j\gamma^\mu D_\mu\psi^j\right\} + \\ &+ i\sqrt{2}gf^{abc}\left(\bar\lambda_a\bar z^j_b L\psi^j_c - \bar\psi^j_a R z^j_b\lambda_c\right) - \frac{1}{2}Yf^{abc}\epsilon_{ijk}\left(\bar\psi^i_a z^j_b L\psi^k_c - \bar\psi^i_a R\bar z^j_b\psi^k_c\right) - \\ &- \frac{1}{2}g^2(f^{abc}\bar z^j_b z^j_c)(f^{ade}\bar z^k_d z^k_e) + \frac{1}{4}Y^2 f^{abc}f^{ade}\epsilon_{ijk}\epsilon_{ilm}z^j_b z^k_c\bar z^l_d\bar z^m_e\end{aligned} \tag{2.1}$$

(L and R are chirality projectors). The theory defined by (2.1) has $\mathcal{N}$=1 supersymmetry. We use separate coupling constants g and Y to distinguish the terms coming from the gauge and superpotential sectors. When $Y = g\sqrt{2}$, SUSY is enhanced to $\mathcal{N}$=4.

Since the manifest symmetry group is now SU(3) $\times$ U(1), we first project onto it the representations of the full SU(4). This can be done by mapping the quantum numbers as

$$[p, q, r] \mapsto [p, q]^{-\frac{1}{2}(p+2q+3r)}\,. \tag{2.2}$$

tation of the work in [6] and [15].

[6] E.g., the order g^4 calculations in [8] were done in the $\mathcal{N}$=2 harmonic superspace formalism.

JHEP11(2001)046

Under this projection, the fermions in the theory are mapped as: $\lambda_{1,2,3} \mapsto \psi_{1,2,3} \in [1,0]^{-\frac{1}{2}}$, $\lambda_4 \mapsto \lambda = [0,0]^{\frac{3}{2}}$, so $\mathbf{4} = [1,0,0] \to [1,0]^{-\frac{1}{2}} \oplus [0,0]^{\frac{3}{2}}$. Similarly the scalars are projected as

$$\mathbf{6} = [0,1,0] \quad \to \quad [1,0]^{1} \oplus [0,1]^{-1} = \{z_j\} \oplus \{\bar{z}^k\} . \tag{2.3}$$

Put more simply, this amounts to rewriting the real scalars ϕ^I, and fermions λ^i as $\phi^i = \frac{1}{\sqrt{2}}(z_i + \bar{z}_i)$, $\phi^{i+3} = \frac{1}{i\sqrt{2}}(z_i - \bar{z}_i)$, and $\lambda^i = \psi_i$, $\lambda^4 = \lambda$. Index $i = 1,2,3$ labels the $\mathbf{3}$ or $\bar{\mathbf{3}}$ of the SU(3) factor of the manifest symmetry group of (2.1).

The R-symmetry group of the theory is SU(4) $\sim$ SO(6), which is a part of the larger superconformal SU(2, 2|4). Unitary representations of $\mathcal{N}$=4 SYM were classified in [1]. As in any conformal theory, operators are classified by their scaling dimension Δ. Each multiplet of SU(2, 2|4) contains an operator of lowest dimension, which is called a primary operator. The action of generators of the conformal group [7] on a primary operator $\Phi(x)$ is given by

$$[P_\mu, \Phi(x)] = i\partial_\mu \Phi(x) \tag{2.4}$$
$$[M_{\mu\nu}, \Phi(x)] = \left[i(x_\mu\partial_\nu - x_\nu\partial_\mu) + \Sigma_{\mu\nu}\right] \Phi(x) \tag{2.5}$$
$$[D, \Phi(x)] = i\left(-\Delta + x^\mu \partial_\mu\right) \Phi(x) \tag{2.6}$$
$$[K_\mu, \Phi(x)] = \left[i(x^2\partial_\mu - 2x_\mu x^\nu \partial_\nu + 2x_\mu \Delta) - 2x^\nu \Sigma_{\mu\nu}\right] \Phi(x) . \tag{2.7}$$

Notice that $[M_{\mu\nu}, \Phi(0)] = \Sigma_{\mu\nu}\Phi(0)$, $[D, \Phi(0)] = -i\Delta\Phi(0)$, and $[K_\mu, \Phi(0)] = 0$. Together with the 16 Poincaré supersymmetry generators Q (and $\bar{Q}$), and 16 special conformal fermionic generators S (and $\bar{S}$), these close in a superconformal algebra of SU(2, 2|4). The additional (anti)commutation relations are schematically given by

$$[D, Q] = -\frac{i}{2}Q , \quad [D, S] = +\frac{i}{2}S , \quad [K, Q] \sim S \quad [P, S] \sim Q , \tag{2.8}$$
$$[Q, S] \sim M + D + R , \quad [S, S] \sim K , \quad [Q_i, Q_j] \sim P\,\delta_{ij}\ (i, j = 1, \ldots, 4) , \tag{2.9}$$

where R stands for the quantum numbers of the R-symmetry group SU(4). The lagrangian of the theory, as well as the action of supersymmetry generators on the elementary fields, are listed in appendix A.

Primary operators of the superconformal group which are annihilated by at least some of the Q-s are called chiral primaries. Descendants of chiral primaries are then chiral operators, in the $\mathcal{N}$=4 sense. Chirality is a property of the whole SU(2, 2|4) multiplet; just being annihilated by say 8 Poincaré SUSY generators doesn't make an operator $\frac{1}{2}$-BPS. Since the supercharges anticommute, we can take a non-chiral operator and act on it with some of the Q-s. The resulting (non-chiral!) operator will be annihilated by the same Q-s.

For a chiral primary field Φ annihilated by a Poincaré supercharge Q, we can write $[Q, \Phi(x)] = 0$ and $[K, \Phi(0)] = 0$, and so $[S, \Phi(0)] \sim [[K, Q], \Phi(0)] = 0$ as well.

[7] See for example [22], or the big review [23], p. 32.

JHEP11(2001)046

Hence we can express the conformal dimension Δ of Φ entirely in terms of its spin Σ and SU(4) quantum numbers R

$$0 = [[Q, S], \Phi(0)] \sim [M + D + R, \Phi(0)] = (\Sigma - i\Delta + R)\,\Phi(0) \tag{2.10}$$

by the superconformal algebra (2.4)–(2.9). Quantum numbers of descendants are related to those of their parent primaries by (2.4)–(2.9) as well. In particular, Δ of any chiral operator can not receive quantum corrections.

3. Gauge invariant scalar composite operators

A kinematic (group theoretic) classification of BPS operators was given in [2]. Chiral primaries [8] are Lorentz scalars, which are made by taking local gauge invariant polynomial combinations of the $\phi^I(x)$, $I = 1, \ldots, 6$. They fall into one of the three families [1]. The simplest one consists of $\frac{1}{2}$-BPS operators. These chiral primaries are annihilated by half of the Q-s, and live in short multiplets with spins ranging from zero to 2. $\frac{1}{2}$-BPS chiral primaries are totally symmetric traceless rank q tensors of the flavor SO(6). SU(4) labels of these representations are $[0, q, 0]$ with the corresponding SO(6) Young tableau [9] $\boxed{}\boxed{\cdots}\boxed{}$, one row of length q. Operators with the highest SU(4) weight in the $[0, q, 0]$ have the form $\mathrm{tr}\,(\phi^1)^q$, modulo the SO(6) traces.[10] Because the color group is SU(N) rather than U(N), $\mathrm{tr}\,\phi^I = 0$ so $q \geq 2$. Conformal dimension of a $\frac{1}{2}$-BPS chiral primary is related to its flavor quantum numbers as $\Delta = q$.

$\frac{1}{4}$-BPS operators form the next simplest family of chiral operators in the classification of [2]. Their multiplets have spins from zero to 3. The primaries belong to $[p, q, p]$ representations, and are annihilated by four out of sixteen Poincaré supercharges. There is a restriction $p \geq 2$: for $p = 0$ the operators are $\frac{1}{2}$-BPS; and in the case $p = 1$, they vanish after we take the SU(N) traces. The highest weight state of $[p, q, p]$ corresponds to the

$$\begin{array}{|c|c|c|c|c|c|}\hline 1 & \cdots & 1 & 1 & \cdots & 1 \\ \hline 2 & \cdots & 2 & \multicolumn{3}{c}{q} \\ \cline{1-3} \multicolumn{3}{c}{p} & \multicolumn{3}{c}{} \end{array} \tag{3.1}$$

SO(6) Young tableau. In the free theory, $\frac{1}{4}$-BPS primaries corresponding to (3.1) are of the form $\mathrm{tr}\,(\phi^1)^{p+q}\,\mathrm{tr}\,(\phi^2)^p$ (modulo (ϕ^1, ϕ^2) antisymmetrizations, and subtraction of the SO(6) traces). However, there are many other ways to partition a given Young tableau, and each may result in a different operator after we take the SU(N) traces. A priori, we do not know if any of them are pure (i.e. eigenstates of the dilatation operator D), or are mixtures of operators with different scaling dimensions. So these

[8] When referring to "primary" fields, we often have in mind the entire SU(4) multiplet to which the actual primary belongs. This slight abuse of notation is common in the literature.

[9] See for example [18] for a general discussion on constructing irreducible tensors of SO(n).

[10] For example, the highest weight state in the [2,0,2] is $\mathrm{tr}\,(\phi^1)^2 - \frac{1}{6}\sum_{I=1}^{6} \mathrm{tr}\,\phi^I\phi^I$. Operators in this representation are usually referred to as "$\mathrm{tr}\,X^2$" in the literature, and are special since their descendants include the SU(4) flavor currents and the stress tensor.

operators should be regarded just as a basis of gauge invariant, local, polynomial, scalar composite operators in the $[p,q,p]$ of SU(4). By taking linear combinations of these, we will construct eigenstates of D in general, and $\frac{1}{4}$-BPS primaries in particular.

For completeness, let us mention the $\frac{1}{8}$-BPS operators, which form the last family of chiral operators in the classification of [2]. $\frac{1}{8}$-BPS multiplets are also short, with spins ranging from zero to 7/2, and the chiral primaries are of the form $\mathrm{tr}\,(\phi^1)^{p+k+q}\mathrm{tr}\,(\phi^2)^{p+k}\mathrm{tr}\,(\phi^3)^k$ (modulo (ϕ^1,ϕ^2,ϕ^3) antisymmetrizations, and minus the SO(6) traces), in the free theory. As before, there is a $k\geq 2$ restriction on the quantum numbers: $k\geq 1$ so the operators are annihilated by exactly two Poincaré supercharges; while operators with $k=1$ necessarily contain commutators after we take the SU(N) traces, as $\mathrm{tr}\,\phi^I=0$. $\frac{1}{8}$-BPS chiral primaries have SU(4) labels $[p,q,p+2k]$, and their scaling dimensions have protected values of $\Delta = 3k+2p+q$. Although these operators are also interesting, we will not study them in this paper.

When calculating n-point functions, it suffices to consider one (nonzero) correlator for a given choice of representations; all others will be related to it by SU(4) Clebsch-Gordon coefficients (by the Wigner-Eckart theorem). Therefore, we are free to take the most convenient representatives of the full SU(4) representations, or of the smaller SU(3) $\times$ U(1) bits into which a given representation of SU(4) breaks down.[11] The combinatorics of the problem simplifies if we consider operators of the form $[(z_1)^{p+q}(z_2)^p]$ and their conjugates, which is what we will do in this paper.

Finally, suppose we have disentangled the mixtures of $[p,q,p]$ scalar composite operators annihilated by a quarter of the Poincaré supercharges, into linear combinations of operators with definite scaling dimension. Furthermore, assume we found an operator $\mathcal{Y}$ whose scaling dimension is protected. Since $\mathcal{Y}$ is a pure operator annihilated by four Poincaré supercharges, it can be either a $\frac{1}{4}$-BPS primary; or a level two descendant of a $\frac{1}{8}$-BPS primary, but this case is excluded [12] by group theory; or a level four descendant of a non-chiral primary. If $\mathcal{Y}$ were non-chiral, its primary would be a scalar composite operator of the form $[z^{2p+q-3}\bar{z}]$; and in all examples that we studied in this paper, such operators do receive $\mathcal{O}(g^2)$ corrections to their scaling dimension.[13] We conclude that a scalar composite operator in the $[p,q,p]$, which is annihilated by a quarter of the supercharges and has a protected scaling dimension $\Delta = 2p+q$, is a $\frac{1}{4}$-BPS chiral primary.

[11] All correlators in the resulting SU(3) $\times$ U(1) representations will have identical spatial dependence, since they come from the same SU(4) representation.

[12] If $\mathcal{Y}$ came from a $\frac{1}{8}$-BPS primary, the parent primary would in the $[p',q',p'+2k]$ representation of SU(4), with $k\geq 2$. On the other hand, to make the scaling dimension and SU(4) Dynkin labels work out right, the only allowed choice is $[p,q,p+2]$, or $k=1$.

[13] But see footnote 1.

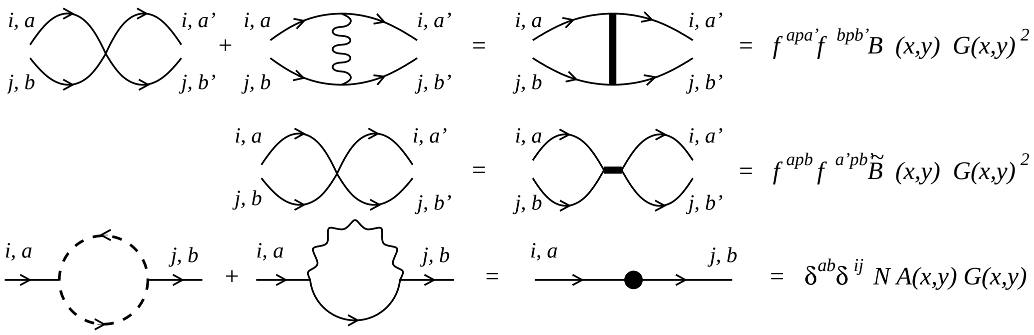


Figure 1: Structures contributing to two-point functions at order g^2. Thick lines correspond to exchanges of the gauge boson and auxiliary fields F_i or D in the $\mathcal{N}$=1 formulation.

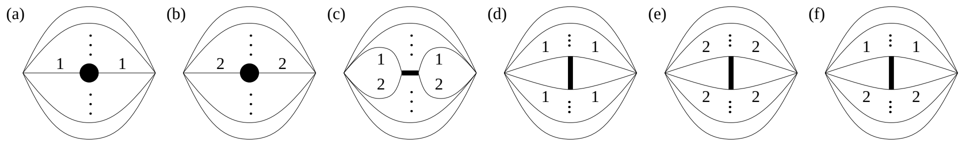


Figure 2: Diagrams contributing to two-point functions of scalars at order g^2.

4. Contributing diagrams

The two point functions we will be calculating in this paper are of the form

$$\langle \left[{z_1}^{(p+q)} {z_2}^p \right](x) \left[\bar{z}_1^{(p+q)} \bar{z}_2^p \right](y) \rangle \,, \tag{4.1}$$

where $[\ldots]$ stands for gauge invariant combinations. The free field part of such a correlator is given by a power of the free scalar propagator $[G(x,y)]^{(2p+q)}$, times a combinatorial factor. At order g^2, there are corrections to the scalar propagator coming from a fermion loop and a gauge boson semi-loop. Also, blocks involving four scalars get contributions from a single gauge boson exchange, and from the four-scalar vertex. Gauge fixing and ghost terms in the lagrangian do not contribute to (4.1) at $\mathcal{O}(g^2)$.

From the lagrangian (2.1) we can read off the structures for the four-scalar blocks, and the leading correction to the propagator. These are shown in figure 1, where they are categorized according to their gauge group (color) index structure (we will use the same notation as in [6]). The scalar propagator remains diagonal in both color and flavor indices at order g^2. Notice that the corrections proportional to $\tilde{B}$ are antisymmetric in i and j, hence they are absent when the scalars in the four legs have the same flavor. Thus we will have to compute contributions of six types[14] (see figure 2).

Most Feynman diagrams we come across are easier to evaluate in position space, where they factorize into products of free propagators and the blocks shown in fig-

[14] If all scalars were of the same flavor, say 1 (as is the case for $\frac{1}{2}$-BPS operators considered in [6]and [15]), we would only have to consider diagrams of types (a) and (d).

JHEP11(2001)046

ure 1, and everything except for the combinatorial factors out front is almost trivial. In momentum space, on the other hand, even the simplest $\mathcal{O}(g^2)$ graphs contain divergent subdiagrams.

The functions A and B will be discussed in detail in section 6. Coordinate dependence of $\tilde{B}$ is parametrically determined by conformal invariance, $\tilde{B}(x,0) = \tilde{a}\log(x^2\mu^2) + \tilde{b}$. The coefficients $\tilde{a}$ and $\tilde{b}$ can be found using, for example, differential regularization [20], or a simpler equivalent prescription: replace $1/x^2 \to 1/(x^2+\epsilon^2)$ for propagators inside integrals ($\epsilon \sim \mu^{-1}$ is related to the renormalization scale). With this,

$$\begin{aligned}\tilde{B}(x,0) &= -\frac{1}{4}Y^2 \int \frac{(d^4z)\,[4\pi^2x^2]^2}{[4\pi^2((z-x)^2+\epsilon^2)]^2\,[4\pi^2(z^2+\epsilon^2)]^2} \\ &= -Y^2\frac{1}{32\pi^2}\left[\log(x^2/\epsilon^2)-1\right]\end{aligned} \tag{4.2}$$

(for $\mathcal{N}$=4 SUSY, $Y^2 = 2g^2$); the same result is obtained in dimensional regularization.

5. The simplest cases

We begin by considering scalar composite operators in representations $[p,q,p]$ of the color SU(4), which have $2p+q = 4$ and 5.

The case of $\Delta = 4 + \mathcal{O}(g^2)$ has been studied before. For example, the authors of [8] argued that there are two operators [15] $\mathcal{O}_1^{[2,0,2]}$ and $\mathcal{O}_2^{[2,0,2]}$, which are made of four scalars and annihilated by four supercharges. $\mathcal{O}_1^{[2,0,2]}$ is a descendant of the Konishi scalar $\left(\sum_{I=1}^{6} \operatorname{tr}\phi^I\phi^I\right)$ and therefore is pure (i.e. is an eigenstate of the dilatation operator), since the Konishi scalar is pure. The other operator, $\mathcal{O}_2^{[2,0,2]}$, contains a piece proportional to $\mathcal{O}_1^{[2,0,2]}$, but the rest is a chiral primary. The method in [8] was to analyze four-point correlators of certain $\frac{1}{2}$-BPS operators, and to look at the possible operators in exchange channels. They found that there is a $\frac{1}{4}$-BPS operator exchanged by demonstrating that there is a pole corresponding to an operator of scaling dimension $\Delta = 4$. They determined this operator to be $\mathcal{Y}_{[2,0,2]} = \mathcal{O}_2^{[2,0,2]} - \frac{4}{N}\mathcal{O}_1^{[2,0,2]}$.

Unfortunately, this method does not generalize to chiral primaries with scaling dimension $\Delta \geq 6$, as we shall see in section 7. So instead we explicitly compute two-point functions of scalar composite operators of a given scaling dimension, and find the ones which do not get corrected. This allows us to fix the normalization of $\frac{1}{4}$-BPS operators as well.

5.1 Scalar composites with weight [2,0,2]

The simplest operators annihilated by four out of sixteen Poincaré supercharges correspond to the highest weight state of the **84** = [2,0,2] of SU(4). The SO(6)

[15] For the notation and definitions, see section 5.1.

JHEP11(2001)046

Young tableau for representation is $\begin{array}{|c|c|}\hline \ & \ \\ \hline \ & \ \\ \hline\end{array}$. An SO(6) irreducible tensor T with this symmetry is made from the corresponding Gl(6) irreducible tensor T^0 by subtracting all possible SO(6) traces:

$$
\begin{aligned}
T_{\begin{array}{|c|c|}\hline a&b\\\hline c&d\\\hline\end{array}} &= T^0_{\begin{array}{|c|c|}\hline a&b\\\hline c&d\\\hline\end{array}} - \frac{1}{4}\Big(T^0_{\begin{array}{|c|c|}\hline \bullet&\bullet\\\hline a&b\\\hline\end{array}}\delta_{cd} + T^0_{\begin{array}{|c|c|}\hline \bullet&\bullet\\\hline c&d\\\hline\end{array}}\delta_{ab} + T^0_{\begin{array}{|c|c|}\hline \bullet&\bullet\\\hline a&d\\\hline\end{array}}\delta_{bc} + T^0_{\begin{array}{|c|c|}\hline \bullet&\bullet\\\hline b&c\\\hline\end{array}}\delta_{ad}\Big) + \\
&\quad + \frac{1}{20}\left(\delta_{ab}\delta_{cd} - \delta_{ad}\delta_{bc}\right) T^0_{\begin{array}{|c|c|}\hline \bullet&\bullet\\\hline \bullet&\bullet\\\hline\end{array}}
\end{aligned}
\tag{5.1}
$$

where $\begin{array}{|c|c|}\hline \bullet&\bullet\\\hline\end{array} = \sum_{a=1}^{6} \begin{array}{|c|c|}\hline a&a\\\hline\end{array}$. Recall that the SU(4) $\to$ SU(3) $\times$ U(1) projection (2.2) is realized on the elementary fields as $\phi_a = \frac{1}{\sqrt{2}}(z_a + \bar{z}_a)$, $\phi_{a+3} = \frac{1}{i\sqrt{2}}(z_a - \bar{z}_a)$, $a = 1, 2, 3$. Under this "3+1 split," the highest weight state of [2,0,2] becomes

$$
\begin{aligned}
T_{\begin{array}{|c|c|}\hline 1&1\\\hline 2&2\\\hline\end{array}} &= \frac{1}{4}\Big(T_{\begin{array}{|c|c|}\hline \mathbf{1}&\mathbf{1}\\\hline \mathbf{2}&\mathbf{2}\\\hline\end{array}} + 2T_{\begin{array}{|c|c|}\hline \bar{\mathbf{1}}&\mathbf{1}\\\hline \mathbf{2}&\mathbf{2}\\\hline\end{array}} + 2T_{\begin{array}{|c|c|}\hline \mathbf{1}&\mathbf{1}\\\hline \bar{\mathbf{2}}&\mathbf{2}\\\hline\end{array}} + 2T_{\begin{array}{|c|c|}\hline \bar{\mathbf{1}}&\mathbf{1}\\\hline \bar{\mathbf{2}}&\mathbf{2}\\\hline\end{array}} + T_{\begin{array}{|c|c|}\hline \bar{\mathbf{1}}&\bar{\mathbf{1}}\\\hline \mathbf{2}&\mathbf{2}\\\hline\end{array}}\Big) + \text{c.c.} \\
&= \frac{1}{4} T^0_{\begin{array}{|c|c|}\hline \mathbf{1}&\mathbf{1}\\\hline \mathbf{2}&\mathbf{2}\\\hline\end{array}} + \text{terms with lower U(1) charge}\,,
\end{aligned}
\tag{5.2}
$$

where in the left hand side, $1 = \phi^1$; and in the right hand side, $\mathbf{1} = z_1$, $\bar{\mathbf{1}} = \bar{z}_1$, etc. We see that after this projection, we don't have to worry about subtracting the SO(6) traces, if we are only interested in the highest U(1) charge operators. Henceforth, we will consider operators made by applying the Young (anti)symmetrizer corresponding to this tableau, to the string of z_i-s.[16]

We can construct one single trace and one double trace operators with the highest [2,0,2] weight. When projected onto SU(3) $\times$ U(1), they become

$$
\begin{aligned}
\mathcal{O}_1^{[2,0,2]} &= \mathrm{tr}\, z_1 z_1 z_2 z_2 - \mathrm{tr}\, z_1 z_2 z_1 z_2 = -\frac{1}{2}\mathrm{tr}\,[z_1, z_2][z_1, z_2] \\
\mathcal{O}_2^{[2,0,2]} &= 2\left(\mathrm{tr}\, z_1 z_1\ \mathrm{tr}\, z_2 z_2 - \mathrm{tr}\, z_1 z_2\ \mathrm{tr}\, z_1 z_2\right)
\end{aligned}
\tag{5.3}
$$

$\mathcal{O}_1^{[2,0,2]}$ is a descendant; by consecutively applying four SUSY transformations,[17] it can be obtained from the Konishi scalar, $\mathcal{K}_0 = \mathrm{tr}\, z_j \bar{z}^j$. More explicitly, acting with the SUSY generator $\bar{Q}_{\bar{\zeta}}$ gives

$$
\delta_{\bar{\zeta}} z_j = 0\,, \quad \delta_{\bar{\zeta}} \bar{z}^j = \sqrt{2}\bar{\zeta}\bar{\psi}^j\,, \quad \delta_{\bar{\zeta}}\bar{\psi}^j = i\epsilon^{jkl}[z_k, z_l]\bar{\zeta}\,,
\tag{5.4}
$$

and with Q_{ζ_3},

$$
\delta_{\zeta_3} z_j = -\sqrt{2}\zeta_3\lambda\delta_{3j}\,, \quad \delta_{\zeta_3}\lambda = 2i[z_1, z_2]\zeta_3\,,
\tag{5.5}
$$

[16] Computing two point functions of operators in representations of SU(3) $\times$ U(1) is as good as computing two point functions of the original SU(4) irreducible operators, see section 2. In the remainder of the paper, we will neglect the SO(6) traces without a comment.

[17] Supersymmetry transformations are listed in appendix A, see equations (A.13)–(A.18).

JHEP11(2001)046

thus

$$
\begin{aligned}
(\bar{Q}_{\bar{\zeta}})^2\mathcal{K}_0 &= \bar{Q}_{\bar{\zeta}}\mathrm{tr}\, z_j\sqrt{2}\bar{\zeta}\bar{\psi}^j = 6i\sqrt{2}(\bar{\zeta}\bar{\zeta})\mathrm{tr}\,[z_1,z_2]z_3\,,\\
(Q_{\zeta_3})^2(\bar{Q}_{\bar{\zeta}})^2\mathcal{K}_0 &= -12i(\bar{\zeta}\bar{\zeta})Q_{\zeta_3}\mathrm{tr}\,[z_1,z_2]\zeta_3\lambda = 24(\bar{\zeta}\bar{\zeta})(\zeta_3\zeta_3)\mathrm{tr}\,[z_1,z_2]^2\\
&= -48(\bar{\zeta}\bar{\zeta})(\zeta_3\zeta_3)\mathcal{O}_1^{[2,0,2]}\,.
\end{aligned}
\tag{5.6}
$$

The four generators which annihilate $\mathcal{O}_1^{[2,0,2]}$, are the ones we acted with to obtain it from $\mathcal{K}_0$. However, since $\mathcal{K}_0$ is a non-chiral primary, its descendant $\mathcal{O}_1^{[2,0,2]}$ is not $\frac{1}{4}$-BPS, despite being annihilated by a quarter of SUSY generators.

The free field results for two point functions of $\mathcal{O}_1^{[2,0,2]}$ and $\mathcal{O}_2^{[2,0,2]}$ are

$$
\begin{pmatrix} \langle\mathcal{O}_1\bar{\mathcal{O}}_1\rangle & \langle\mathcal{O}_1\bar{\mathcal{O}}_2\rangle \\ \langle\mathcal{O}_2\bar{\mathcal{O}}_1\rangle & \langle\mathcal{O}_2\bar{\mathcal{O}}_2\rangle \end{pmatrix}_{\mathrm{free}} = \frac{3(N^2-1)G^4}{16}\begin{pmatrix} N^2 & 4N \\ 4N & 8(N^2-2)\end{pmatrix}
\tag{5.7}
$$

while the leading corrections are found to be

$$
\begin{pmatrix} \langle\mathcal{O}_1\bar{\mathcal{O}}_1\rangle & \langle\mathcal{O}_1\bar{\mathcal{O}}_2\rangle \\ \langle\mathcal{O}_2\bar{\mathcal{O}}_1\rangle & \langle\mathcal{O}_2\bar{\mathcal{O}}_2\rangle \end{pmatrix}_{g^2} = \frac{9(N^2-1)G^4(\tilde{B}N)}{16}\begin{pmatrix} N^2 & 4N \\ 4N & 16\end{pmatrix};
\tag{5.8}
$$

here $\langle\mathcal{O}_i\bar{\mathcal{O}}_j\rangle \equiv \langle\mathcal{O}_i^{[2,0,2]}(x)\bar{\mathcal{O}}_j^{[2,0,2]}(y)\rangle$, and $G \equiv G(x,y)$, $\tilde{B} \equiv \tilde{B}(x,y)$. Some helpful formulae we used for deriving (5.7) and (5.8) are collected in appendix B.

By looking at (5.8), we conclude that neither $\mathcal{O}_1^{[2,0,2]}$ nor $\mathcal{O}_2^{[2,0,2]}$ are chiral. However, there is a linear combination of these two operators which has protected two point functions at order g^2. The operator

$$
\mathcal{Y}_{[2,0,2]}(x) \equiv \mathcal{O}_2^{[2,0,2]}(x) - \frac{4}{N}\mathcal{O}_1^{[2,0,2]}(x)
\tag{5.9}
$$

satisfies $\langle\mathcal{Y}\bar{\mathcal{Y}}\rangle = \langle\mathcal{Y}\bar{\mathcal{O}}_1\rangle = 0$, so $\mathcal{Y}_{[2,0,2]}$ is orthogonal to the descendant of the Konishi operator $\mathcal{O}_1^{[2,0,2]}$, and has protected dimension $\Delta_{\mathcal{Y}} = 4$ at order g^2. Computationally, this cancellation is rather intricate: all $\langle\mathcal{O}_i\bar{\mathcal{O}}_j\rangle$ have very different large N behavior.

We can also calculate the two point function of the Konishi scalar with itself:

$$
\langle\mathcal{K}_0(x)\bar{\mathcal{K}}_0(y)\rangle = 3(N^2-1)[G(x,y)]^2\left\{1+3\tilde{B}(x,y)N+\mathcal{O}(g^4)\right\}
\tag{5.10}
$$

so $\langle\mathcal{O}_1^{[2,0,2]}(x)\bar{\mathcal{O}}_1^{[2,0,2]}(y)\rangle = \frac{1}{16}N^2[G(x,y)]^2\langle\mathcal{K}_0(x)\bar{\mathcal{K}}_0(y)\rangle + \mathcal{O}(g^4)$, which is just the free theory result. In particular, $\mathcal{K}_0$ and its descendant $\mathcal{O}_1^{[2,0,2]}$ have the same normalization and their scaling dimensions differ by 2, as they must: with $\tilde{B}(x,0)$ given by (4.2), the scaling dimension of $\mathcal{O}_1^{[2,0,2]}$ is $\Delta_1 = 4 + \frac{3g^2N}{16\pi^2} + \mathcal{O}(g^4)$, and that of $\mathcal{K}_0$ is $\Delta_{\mathcal{K}} = 2 + \frac{3g^2N}{16\pi^2} + \mathcal{O}(g^4)$, in agreement with [8]. The mixture $\mathcal{O}_2^{[2,0,2]} = \mathcal{Y}_{[2,0,2]} + \frac{4}{N}\mathcal{O}_1^{[2,0,2]}$ has the two-point function with itself which breaks down into two pieces,

$$
\begin{aligned}
\langle\mathcal{O}_2^{[2,0,2]}(x)\bar{\mathcal{O}}_2^{[2,0,2]}(y)\rangle &= \langle\mathcal{Y}_{[2,0,2]}(x)\bar{\mathcal{Y}}_{[2,0,2]}(y)\rangle + \frac{16}{N^2}\langle\mathcal{O}_1^{[2,0,2]}(x)\bar{\mathcal{O}}_1^{[2,0,2]}(y)\rangle\\
&= \frac{C_{\mathcal{Y}}}{(x-y)^{2\Delta_{\mathcal{Y}}}} + \frac{16}{N^2}\frac{C_1}{(x-y)^{2\Delta_1}}\,.
\end{aligned}
$$

Logarithmic corrections to $\langle \mathcal{O}_2^{[2,0,2]}(x)\bar{\mathcal{O}}_2^{[2,0,2]}(y)\rangle$ are due entirely to the second term $\frac{16}{N^2}\langle \mathcal{O}_1^{[2,0,2]}(x)\bar{\mathcal{O}}_1^{[2,0,2]}(y)\rangle$.

$\mathcal{O}(g^2)$ corrections to all two-point functions just considered are proportional to $\tilde{B}$. In other words, only the contributions due to diagrams of type (c) in figure 2 survive, and all other corrections cancel. As we will show in section 6, this is a general phenomenon. Also, in the large N limit the fraction $\langle \mathcal{O}_1\bar{\mathcal{O}}_2\rangle/\langle \mathcal{O}_1\bar{\mathcal{O}}_1\rangle$ is suppressed; it vanishes in the limit $N\to\infty$, g^2N fixed.[18]

One can show that conformal dimension of $\mathcal{Y}_{[2,0,2]}$ is protected perturbatively (to order g^4) and nonperturbatively (for any instanton number) as well, see [8]. Non-renormalization of scaling dimension of $\mathcal{Y}$ hints at its BPS property. Following the authors of [8], we suggest that it is indeed a $\frac{1}{4}$-BPS chiral primary operator.

5.2 Scalar composites with weight [2,1,2]

The story is similar in the case of the **300** = [2,1,2] of SU(4). The only scalar composite operators corresponding to the highest weight of this representation are (after the projection onto SU(3) $\times$ U(1), as discussed in sections 2 and 5.1)

$$\begin{aligned} \mathcal{O}_1^{[2,1,2]} &= \mathrm{tr}\, z_1z_1z_1z_2z_2 - \mathrm{tr}\, z_1z_1z_2z_1z_2 = -\frac{1}{2}\mathrm{tr}\,[z_1,z_2][z_1^2,z_2] \\ \mathcal{O}_2^{[2,1,2]} &= \mathrm{tr}\, z_1z_1z_1\ \mathrm{tr}\, z_2z_2 - 2\mathrm{tr}\, z_1z_2\ \mathrm{tr}\, z_1z_2z_1 + \mathrm{tr}\, z_1z_2z_2\ \mathrm{tr}\, z_1z_1 \end{aligned} \tag{5.11}$$

$\mathcal{O}_1^{[2,1,2]}$ is a descendant; $(Q_{\zeta_3})^2(\bar{Q}_{\bar{\zeta}})^2\mathrm{str}\, z_1z_i\bar{z}^i \propto \mathcal{O}_1^{[2,1,2]}$ just like in equation (5.6). Born level and order g^2 two point functions are

$$\begin{pmatrix} \langle \mathcal{O}_1\bar{\mathcal{O}}_1\rangle & \langle \mathcal{O}_1\bar{\mathcal{O}}_2\rangle \\ \langle \mathcal{O}_2\bar{\mathcal{O}}_1\rangle & \langle \mathcal{O}_2\bar{\mathcal{O}}_2\rangle \end{pmatrix} = \frac{(N^2-1)(N^2-4)G^5}{16N}\left\{\begin{pmatrix} N^2 & 6N \\ 6N & 6(N^2-3)\end{pmatrix} + \right. \tag{5.12}$$
$$\left. + 4\tilde{B}N\begin{pmatrix} N^2 & 6N \\ 6N & 36\end{pmatrix} + \mathcal{O}(g^4)\right\},$$

where $\langle \mathcal{O}_i\bar{\mathcal{O}}_j\rangle \equiv \langle \mathcal{O}_i^{[2,1,2]}(x)\bar{\mathcal{O}}_j^{[2,1,2]}(y)\rangle$, and $G \equiv G(x,y)$, $\tilde{B} \equiv \tilde{B}(x,y)$ as before. (Note that corrections proportional to A and B cancel again.) There is a linear combination of $\mathcal{O}_1^{[2,1,2]}$ and $\mathcal{O}_2^{[2,1,2]}$

$$\mathcal{Y}_{[2,1,2]}(x) \equiv \mathcal{O}_2^{[2,1,2]}(x) - \frac{6}{N}\mathcal{O}_1^{[2,1,2]}(x) \tag{5.13}$$

whose two point functions with arbitrary operators do not receive perturbative order g^2 corrections. Again, it seems reasonable to conclude that $\mathcal{Y}_{[2,1,2]}$ is a $\frac{1}{4}$-BPS operator, as it is annihilated by four out of sixteen supercharges, has a protected scaling dimension $\Delta_{\mathcal{Y}} = 5$ (at order g^2), and contains no descendant pieces, $\langle \mathcal{Y}_{[2,1,2]}(x)\bar{\mathcal{O}}_1^{[2,1,2]}(y)\rangle = 0$.

[18] The large N limit will be analyzed in more detail in section 9.

JHEP11(2001)046

6. A gauge invariance argument

In sections 5.1 and 5.2, we explicitly calculated the $\mathcal{O}(g^2)$ corrections to two point functions of scalar composite operators. We found that corrections proportional to A and B cancel, i.e. gauge group combinatorics demands that diagrams containing a gauge boson exchange do not arise in the correlator. Here we give a general derivation of this fact, which boils down to gauge invariance of the operators in question, and gauge dependence of A and B.

The two point functions we have been considering are of the form

$$\langle [z^m](x)\; [\bar{z}^m](y) \rangle \,, \tag{6.1}$$

where $[z^m](x)$ is some gauge-invariant homogeneous polynomial (of degree m) in the z_i^a-s. Diagrams involving a gauge boson exchange which contribute to the two-point functions of the form (6.1), are proportional to either $A(x,y)$ or $B(x,y)$, see figure 1. By using nonrenormalization of the two point function $\langle \mathrm{tr}\, z_1 z_2(x)\ \mathrm{tr}\, \bar{z}_1\bar{z}_2(y)\rangle$, one can immediately see [6] that $B(x,y) = -2A(x,y)$, so these contributions add up to

$$\langle [z^m]\,(x)\,[\bar{z}^m]\,(y)\rangle_{(A+B)} = c_g A(x,y)[G(x,y)]^m \tag{6.2}$$

where c_g is some combinatorial coefficient.

Conformal invariance restricts $A(x,y) = a\log x^2\mu^2 + b$. The constants a and b turn out to be gauge dependent. The gauge fixing parameter ξ enters the expression for the scalar propagator as[19]

$$\begin{aligned} &= \xi g^2 \mu^{4-d}\frac{1}{p^4}\int \frac{(d^dk)}{(2\pi)^d}\frac{[(2p+k)\cdot k]^2}{k^4(p+k)^2} + (\xi\text{-independent}) \\ &= \xi g^2 \tfrac{1}{2}\pi^{2+\epsilon}\frac{1}{(p^2)^{1-\epsilon}\mu^{2\epsilon}}\left[\frac{1}{\epsilon}+\gamma+\mathcal{O}(\epsilon)\right] + (\xi\text{-independent}) \end{aligned} \tag{6.3}$$

in momentum space (in dimensional regularization [16]; $\epsilon = \frac{d}{2}-2$ and γ is Euler's gamma constant), so in position space

$$\begin{aligned} A(x,0) &= \frac{1}{2}\pi^2 g^2 \xi\left[\log x^2\mu^2 + \log 4\pi - \gamma\right] + (\xi\text{-independent}) \\ &\equiv a\log x^2\mu^2 + b \end{aligned} \tag{6.4}$$

after factoring out the free propagator $G(x,0)$. Both a and b have pieces linear in the gauge fixing parameter ξ.

[19] By changing ξ, we can vary both the pole piece and the $\mathcal{O}(\epsilon^0)$ term (but we can't make them both zero simultaneously; the ξ-independent part is proportional to a different integral). Compare this with [19], where order g^2 corrections to the scalar propagators were found to vanish in super-Feynman gauge of $\mathcal{N}$=1 formulation of the theory.

JHEP11(2001)046

Since a correlator of gauge invariant operators must be gauge independent, the combinatorial coefficient multiplying $A(x,y)$ in equation (6.2) must vanish; we necessarily have $c_g = 0$. This is a general phenomenon, illustrated by an explicit calculation of sections 5.1 and 5.2: gauge dependent contributions are proportional to $2A + B = 0$.

In the $\mathcal{O}(g^2)$ calculations of correlators of $\frac{1}{2}$-BPS operators [6] and [15], there were no other contributions to two-point functions except for the ones proportional to A and B. Thus, gauge invariance together with $\mathcal{N}$=4 SUSY (which is needed to make $2A + B = 0$) guarantees that the correlators of [6] and [15] receive no order g^2 corrections.

7. Operators of dimension 6 and higher

At this point, we would like to consider operators made of $2p + q \geq 6$ scalar fields. According to the classification of [2], these $[p, q, p]$ operators are the candidates for $\frac{1}{4}$-BPS chiral primaries. However, a new complication arises compared to the cases of $2p+q \leq 5$ studied in sections 5.1 and 5.2. Now, there are many ways in which we can make gauge invariant combinations of fields, and hence many scalar composites have to be taken into account. Apart from single and double trace operators we have seen so far, operators made of three or more traces also have to be considered.

This phenomenon has a counterpart in the context of $\frac{1}{2}$-BPS operators, see [15]. The crucial difference is that in our case, none of the scalar composites are pure, and only some special mixtures have a well defined scaling dimension. In general, the "naive" scalar composite operators will have nonvanishing two point functions with each other, whenever this is allowed by group theory. Neither is it easy to extract the descendant operators. Unlike in the simplest cases of section 5, operators containing commutators are not pure, but contain pieces which are descendants of different operators.

Thus the prescription of sections 5.1 and 5.2 (find the pure non-BPS primaries, list their descendants, then subtract these pieces from the candidate $\frac{1}{4}$-BPS operator) no longer goes through; we need to do something else. So instead we calculate the two point functions of highest weight $[p, q, p]$ scalar composites $\mathcal{O}_i^{[p,q,p]}$, and arrange them as[20]

$$\langle \mathcal{O}_i^{[p,q,p]}(x) \bar{\mathcal{O}}_j^{[p,q,p]}(y) \rangle \equiv [G(x,y)]^{(2p+q)} \left[\mathbf{F}_{ij} + \tilde{B}(x,y) N \, \mathbf{G}_{ij} + \mathcal{O}(g^4) \right] \tag{7.1}$$

with $\mathbf{F}$ the matrix of combinatorial factors at free level, and $\mathbf{G}$, of order g^2 correction combinatorial factors. Note that there can be no corrections proportional to A or B, as was argued in section 6. Both $\mathbf{F}$ and $\mathbf{G}$ are matrices of pure numbers; they

[20] The operators we are working with are after the projection onto $\mathrm{SU}(3) \times \mathrm{U}(1)$, as discussed in sections 2 and 5.1. The $\mathcal{O}_i^{[p,q,p]}$ are made of only z-s and no $\bar{z}$-s.

are still functions of N, but coordinate and g^2 dependence are all absorbed in $\tilde{B}$ and $[G(x,y)]^{(2p+q)}$. Now the problem becomes one of linear algebra: starting with a basis of $\mathcal{O}_i^{[p,q,p]}$, we want to find their linear combinations $\mathcal{Y}_j^{[p,q,p]}$ that are pure operators. The $\mathcal{Y}_j^{[p,q,p]}$ have a well defined renormalized scaling dimension $\Delta_j = \Delta^0 + \Delta_j^1 + \mathcal{O}(g^4)$; $\Delta^0 = 2p+q$ for all $\mathcal{O}_i^{[p,q,p]}$ and hence for all $\mathcal{Y}_j^{[p,q,p]}$. Such operators can be chosen orthogonal at Born level, and so

$$\langle \mathcal{Y}_i^{[p,q,p]} \bar{\mathcal{Y}}_j^{[p,q,p]} \rangle = \frac{C_i^{[p,q,p];0}\delta_{ij}}{x^{2\Delta^0}} \left[1 + \beta_i - \Delta_i^1 \log \mu^2 x^2 + \mathcal{O}(g^4)\right] \tag{7.2}$$

to order g^2. Coefficients $\Delta_j^1 \sim \beta_j \sim g^2$ correspond to corrections of $\mathcal{Y}_j^{[p,q,p]}$'s scaling dimension and its normalization; β_j depends on the renormalization scale μ. To distinguish the pure operators which do receive corrections to their scaling dimension, we will denote them by $\tilde{\mathcal{Y}}$, and reserve the notation $\mathcal{Y}$ for the ones that have $\mathcal{O}(g^2)$ protected two point functions.

This is a standard problem, analogous to finding the normal modes of small oscillations of a mechanical system (see [17], for example). We have to diagonalize a symmetric matrix $\mathbf{G}$ of corrections with respect to the symmetric positive definite matrix $\mathbf{F}$ of free correlators. In other words, we need to find the eigenvalues of matrix $\mathbf{F}^{-1}\mathbf{G}$. If some of them vanish, the corresponding eigenvectors are operators whose two point functions (with themselves as well as with other operators) do not get order g^2 corrections. We conjecture that these are in fact the $\frac{1}{4}$-BPS operators we are after.

7.1 Scalar composites with weight [2,2,2]

Operators with $\Delta^0 = 6$ are the lowest dimension operators which illustrate the new phenomenon. Corresponding to the highest weight of the **729** = [2,2,2] of SU(4), there are five linearly independent[21] operators:

one single trace operator

$$\mathcal{O}_1^{[2,2,2]} \equiv \mathrm{tr}\, z_1 z_1 z_1 z_1 z_2 z_2 - \frac{2}{3}\, \mathrm{tr}\, z_1 z_1 z_1 z_2 z_1 z_2 - \frac{1}{3}\, \mathrm{tr}\, z_1 z_1 z_2 z_1 z_1 z_2 \tag{7.3}$$

three double trace operators

$$\begin{aligned}
\mathcal{O}_2^{[2,2,2]} \equiv\ & \mathrm{tr}\, z_1 z_1 z_1 z_1\ \mathrm{tr}\, z_2 z_2 - 2\, \mathrm{tr}\, z_1 z_1 z_1 z_2\ \mathrm{tr}\, z_1 z_2 + \\
& + \frac{1}{3}\left(2\, \mathrm{tr}\, z_1 z_1 z_2 z_2 + \mathrm{tr}\, z_1 z_2 z_1 z_2\right)\ \mathrm{tr}\, z_1 z_1 \\
\mathcal{O}_3^{[2,2,2]} \equiv\ & \mathrm{tr}\, z_1 z_1 z_1\ \mathrm{tr}\, z_1 z_2 z_2 - \mathrm{tr}\, z_1 z_1 z_2\ \mathrm{tr}\, z_1 z_1 z_2 \\
\mathcal{O}_4^{[2,2,2]} \equiv\ & \frac{1}{2} \mathrm{tr}\, z_1 z_1 z_1 z_1\ \mathrm{tr}\, z_2 z_2 - \mathrm{tr}\, z_1 z_1 z_1 z_2\ \mathrm{tr}\, z_1 z_2 + \\
& + \frac{1}{2}\left(4\, \mathrm{tr}\, z_1 z_1 z_2 z_2 - 3\, \mathrm{tr}\, z_1 z_2 z_1 z_2\right)\ \mathrm{tr}\, z_1 z_1
\end{aligned} \tag{7.4}$$

and one triple trace operator

[21] For $N \leq 4$, the number of independent gauge invariant operators is smaller.

$$\mathcal{O}_5^{[2,2,2]} \equiv \mathrm{tr}\, z_1 z_1 \left(\mathrm{tr}\, z_1 z_1 \ \mathrm{tr}\, z_2 z_2 - \mathrm{tr}\, z_1 z_2 \ \mathrm{tr}\, z_1 z_2\right) \tag{7.5}$$

The $\mathcal{O}_i^{[2,2,2]}$ are constructed by applying the proper Young operator to all possible gauge invariant combinations of $(z_1)^4(z_2)^2$. The Young operator corresponds to the tableau $\yng(4,2)$, while gauge invariant combinations amount to grouping the z_i-s into traces.[22] None of the $\mathcal{O}_{1,\ldots,5}^{[2,2,2]}$ have a well defined scaling dimension.

So we calculate explicitly the $\frac{1}{2} \cdot 5 \cdot (5+1) = 15$ two point functions of [2,2,2] operators, and arrange them as in equation (7.1). We calculate the matrix $\mathbf{F}$ of free correlator combinatorial factors; and the matrix $\mathbf{G}$, of order g^2 correction combinatorial factors (the expressions are not transparent so we list them in appendix C.1). It turns out that two eigenvalues of $\mathbf{F}^{-1}\mathbf{G}$ vanish, signaling $\frac{1}{4}$-BPS operators; the other three eigenvalues satisfy a cubic and so can be easily computed for general N. The two linear combinations of operators which satisfy $\langle \mathcal{Y}_j^{[2,2,2]} \bar{\mathcal{O}}_i^{[2,2,2]} \rangle_{g^2} = 0$ for all i, are

$$\mathcal{Y}_1^{[2,2,2]} = -\frac{8N}{(N^2-4)} \mathcal{O}_1^{[2,2,2]} + \mathcal{O}_2^{[2,2,2]} + \frac{8}{3\,(N^2-4)} \left(2\mathcal{O}_3^{[2,2,2]} + \mathcal{O}_4^{[2,2,2]}\right) \tag{7.6}$$

and the one orthogonal to it (in the sense that $\langle \mathcal{Y}_1^{[2,2,2]}(x) \bar{\mathcal{Y}}_2^{[2,2,2]}(y) \rangle = 0$)

$$\begin{aligned} \mathcal{Y}_2^{[2,2,2]} =& \frac{144\,(N^2-4)\,(N^2-2)}{3N^6 - 47N^4 + 248N^2 - 192} \mathcal{O}_1^{[2,2,2]} - \frac{3N\,(N^2-7)\,(3N^2+8)}{3N^6 - 47N^4 + 248N^2 - 192} \mathcal{O}_2^{[2,2,2]} - \\ & - \frac{2N\,(3N^4 - 23N^2 + 104)}{3N^6 - 47N^4 + 248N^2 - 192} \left(2\mathcal{O}_3^{[2,2,2]} + \mathcal{O}_4^{[2,2,2]}\right) + \mathcal{O}_5^{[2,2,2]} \end{aligned} \tag{7.7}$$

These are the only candidates for $\frac{1}{4}$-BPS operators in representation [2,2,2]. We see that $\mathcal{Y}_1^{[2,2,2]}$ is constructed out of double and single trace operators, while $\mathcal{Y}_2^{[2,2,2]}$ is made up of everything that is available. Both (7.6) and (7.7) are exact in N and are not large N approximations.

The remaining three pure operators involve mixtures of all the triple, double, and single trace operators $\mathcal{O}_{1,\ldots,5}^{[2,2,2]}$. The coefficients are nondescript irrational (unlike for $\mathcal{Y}_1^{[2,2,2]}$ and $\mathcal{Y}_2^{[2,2,2]}$) functions of N. In the large N limit, another operator appears which has a vanishing correction to its two point functions with all $\mathcal{O}_i$-s.

7.2 Scalar composites with weight [3,1,3]

Two $[p,q,p]$ representations have $2p+q=7$. These are [3,1,3] = **960** and [2,3,2] = **1470**. In the first case, the scalar composite operators are

one single trace

[22] We will give a more explicit discussion about how the fields are grouped into traces, when we talk about [2,3,2] operators in section 7.3 (equations 7.24-7.27).

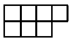
$$
\begin{aligned}
\mathcal{O}_1^{[3,1,3]} \equiv\ & \frac{1}{3}\mathrm{tr}\, z_1z_1z_1z_1z_2z_2z_2 - \frac{1}{2}\mathrm{tr}\, z_1z_1z_1z_2z_1z_2z_2 - \frac{1}{2}\mathrm{tr}\, z_1z_1z_1z_2z_2z_1z_2 + \\
& + \frac{1}{3}\mathrm{tr}\, z_1z_1z_2z_1z_2z_1z_2 + \frac{1}{3}\mathrm{tr}\, z_1z_1z_2z_2z_1z_1z_2
\end{aligned}
\tag{7.8}
$$

two double trace operators

$$
\begin{aligned}
\mathcal{O}_2^{[3,1,3]} \equiv\ & \mathrm{tr}\, z_1z_1z_1z_1\ \mathrm{tr}\, z_2z_2z_2 - 3\,\mathrm{tr}\, z_1z_1z_1z_2\ \mathrm{tr}\, z_1z_2z_2 + \\
& + \left(2\,\mathrm{tr}\, z_1z_1z_2z_2 + \mathrm{tr}\, z_1z_2z_1z_2\right)\ \mathrm{tr}\, z_1z_1z_2 - \mathrm{tr}\, z_1z_2z_2z_2\ \mathrm{tr}\, z_1z_1z_1 \\
\mathcal{O}_3^{[3,1,3]} \equiv\ & -\left(\mathrm{tr}\, z_1z_1z_2z_2z_2 - \mathrm{tr}\, z_1z_2z_1z_2z_2\right)\ \mathrm{tr}\, z_1z_1 + \\
& + \left(\mathrm{tr}\, z_1z_1z_1z_2z_2 - \mathrm{tr}\, z_1z_1z_2z_1z_2\right)\ \mathrm{tr}\, z_1z_2
\end{aligned}
\tag{7.9}
$$

and one triple trace operator

$$
\begin{aligned}
\mathcal{O}_4^{[3,1,3]} \equiv\ & \mathrm{tr}\, z_1z_2\left(2\,\mathrm{tr}\, z_1z_2\ \mathrm{tr}\, z_1z_2z_2 - \mathrm{tr}\, z_2z_2\ \mathrm{tr}\, z_1z_1z_1 - 3\mathrm{tr}\, z_1z_1\ \mathrm{tr}\, z_1z_2z_2\right) + \\
& + \mathrm{tr}\, z_1z_1\left(\mathrm{tr}\, z_2z_2\ \mathrm{tr}\, z_1z_1z_2 + \mathrm{tr}\, z_1z_1\ \mathrm{tr}\, z_2z_2z_2\right)
\end{aligned}
\tag{7.10}
$$

The operators are obtained in the same way as before: the Young operator corresponds to the diagram  of SO(6), and the partitions are 7=7, 4+3, 5+2, and 2+2+3. Operators resulting from other partitions turn out to be linear combinations of the $\mathcal{O}_i^{[3,1,3]}$ above. We see that even the number of linearly independent scalar composites for a given Dynkin label is a complicated function of the scaling dimension Δ. Like in the [2,2,2] case, none of the $\mathcal{O}_i^{[3,1,3]}$ are pure. $\mathcal{O}_1^{[3,1,3]}$ and $\mathcal{O}_3^{[3,1,3]}$ contain commutators, and are likely to be linear combinations of descendants of non-BPS primaries.

The matrix $\mathbf{F}^{-1}\mathbf{G}$ is now 4×4; it has two zero eigenvalues, while the other two satisfy a quadratic equation. The formulae are more tractable than in the [2,2,2] case, so we present some of the details here. We find

$$
\frac{768}{5N^3\left(N^2-1\right)\left(N^2-4\right)}\mathbf{F} = \begin{pmatrix}
\frac{N^2+3}{N^2} & \frac{12}{N} & -\frac{12}{N} & \frac{72}{N^2} \\
\frac{12}{N} & \frac{36\left(N^4-8N^2+18\right)}{N^4} & -\frac{108}{N^2} & \frac{72\left(2N^2-9\right)}{N^3} \\
\frac{-12}{N} & -\frac{108}{N^2} & \frac{36\left(N^2+6\right)}{5N^2} & -\frac{72}{N} \\
\frac{72}{N^2} & \frac{72\left(2N^2-9\right)}{N^3} & -\frac{72}{N} & \frac{72\left(N^2-3\right)}{N^2}
\end{pmatrix}
\tag{7.11}
$$

for the matrix of free combinatorial factors, and

$$
\begin{aligned}
& \frac{128}{25N^3\left(N^2-1\right)\left(N^2-4\right)}\mathbf{G} = \\
& \qquad = \begin{pmatrix}
\frac{N^2+7}{N^2} & \frac{12\left(N^2+3\right)}{N^3} & -\frac{72\left(N^2+1\right)}{5N^3} & \frac{96}{N^2} \\
\frac{12\left(N^2+3\right)}{N^3} & \frac{144}{N^2} & -\frac{144}{N^2} & \frac{864}{N^3} \\
-\frac{72\left(N^2+1\right)}{5N^3} & -\frac{144}{N^2} & \frac{144\left(N^2+16\right)}{25N^2} & -\frac{288\left(N^2+6\right)}{5N^3} \\
\frac{96}{N^2} & \frac{864}{N^3} & -\frac{288\left(N^2+6\right)}{5N^3} & \frac{576}{N^2}
\end{pmatrix}
\end{aligned}
\tag{7.12}
$$

for the matrix of corrections proportional to $\tilde{B}(x,y)N$.

The vectors killed by $\mathbf{F}^{-1}\mathbf{G}$ work out to be

$$\mathcal{Y}_1^{[3,1,3]} = -\frac{12N}{N^2-2}\,\mathcal{O}_1^{[3,1,3]} + \mathcal{O}_2^{[3,1,3]} - \frac{5}{N^2-2}\,\mathcal{O}_3^{[3,1,3]} \tag{7.13}$$

$$\mathcal{Y}_2^{[3,1,3]} = \frac{96}{N^2-4}\mathcal{O}_1^{[3,1,3]} - \frac{4N}{N^2-4}\mathcal{O}_2^{[3,1,3]} + \frac{10N}{N^2-4}\mathcal{O}_3^{[3,1,3]} + \mathcal{O}_4^{[3,1,3]} \tag{7.14}$$

They correspond to zero eigenvalues of $\mathbf{F}^{-1}\mathbf{G}$, and so are the candidates for $\frac{1}{4}$-BPS primaries in the [3,1,3].[23] The remaining eigenvectors of $\mathbf{F}^{-1}\mathbf{G}$ are

$$\tilde{\mathcal{Y}}_3^{[3,1,3]} = \mathcal{O}_1^{[3,1,3]} - \frac{10}{3\left(N+\sqrt{N^2+160}\right)}\mathcal{O}_3^{[3,1,3]} \tag{7.15}$$

$$\tilde{\mathcal{Y}}_4^{[3,1,3]} = \mathcal{O}_3^{[3,1,3]} - \frac{3\left(N-\sqrt{N^2+160}\right)}{10}\mathcal{O}_1^{[3,1,3]} \tag{7.16}$$

corresponding to the eigenvalues $27+\frac{3\sqrt{160+N^2}}{N}$ for $\tilde{\mathcal{Y}}_3^{[3,1,3]}$, and $27-\frac{3\sqrt{160+N^2}}{N}$ for $\tilde{\mathcal{Y}}_4^{[3,1,3]}$. Expressions (7.14-7.16) are exact in N, and are not just large N approximations. As expected, the descendants $\tilde{\mathcal{Y}}$ are mixtures of operators involving commutators.

Note that both the g^2 corrections to the scaling dimension of $\tilde{\mathcal{Y}}$ and their expansion coefficients involve radicals (so there is really no way to "guess" the pure primaries such operators came from). Also, radiative corrections to all Δ-s are non-negative, since at free field level the $\mathcal{O}_i^{[3,1,3]}$ are annihilated by a quarter of the supercharges, and hence saturate the BPS bound.

7.3 Scalar composites with weight [2,3,2]

The other $[p,q,p]$ of SU(4) with $2p+q=7$ is the $[2,3,2]=\mathbf{1470}$. Here we have seven linearly independent operators corresponding to the highest weight state:

one single trace operator

$$\mathcal{O}_1^{[2,3,2]} \equiv 2\,\mathrm{tr}\,z_1z_1z_1z_1z_1z_2z_2 - \mathrm{tr}\,z_1z_1z_1z_1z_2z_1z_2 - \mathrm{tr}\,z_1z_1z_1z_2z_1z_1z_2 \tag{7.17}$$

four double trace operators

$$\mathcal{O}_2^{[2,3,2]} \equiv 2\,\mathrm{tr}\,z_1z_1z_1z_1z_1\ \mathrm{tr}\,z_2z_2 - 4\,\mathrm{tr}\,z_1z_1z_1z_1z_2\ \mathrm{tr}\,z_1z_2 + \left(\mathrm{tr}\,z_1z_1z_1z_2z_2 + \mathrm{tr}\,z_1z_1z_2z_1z_2\right)\ \mathrm{tr}\,z_1z_1 \tag{7.18}$$

$$\mathcal{O}_3^{[2,3,2]} \equiv \mathrm{tr}\,z_1z_1z_1z_1z_1\ \mathrm{tr}\,z_2z_2 - 2\,\mathrm{tr}\,z_1z_1z_1z_1z_2\ \mathrm{tr}\,z_1z_2 + \left(8\,\mathrm{tr}\,z_1z_1z_1z_2z_2 - 7\,\mathrm{tr}\,z_1z_1z_2z_1z_2\right)\ \mathrm{tr}\,z_1z_1 \tag{7.19}$$

$$\mathcal{O}_4^{[2,3,2]} \equiv 3\,\mathrm{tr}\,z_1z_1z_1z_1\ \mathrm{tr}\,z_1z_2z_2 - 6\,\mathrm{tr}\,z_1z_1z_1z_2\ \mathrm{tr}\,z_1z_1z_2 + \left(2\,\mathrm{tr}\,z_1z_1z_2z_2 + \mathrm{tr}\,z_1z_2z_1z_2\right)\ \mathrm{tr}\,z_1z_1z_1 \tag{7.20}$$

$$\mathcal{O}_5^{[2,3,2]} \equiv 3\,\mathrm{tr}\,z_1z_1z_1z_1\ \mathrm{tr}\,z_1z_2z_2 - 6\,\mathrm{tr}\,z_1z_1z_1z_2\ \mathrm{tr}\,z_1z_1z_2 + \left(7\,\mathrm{tr}\,z_1z_1z_2z_2 - 4\,\mathrm{tr}\,z_1z_2z_1z_2\right)\ \mathrm{tr}\,z_1z_1z_1 \tag{7.21}$$

and two triple trace operators

[23] We chose $\mathcal{Y}_2^{[3,1,3]}$ to be orthogonal to $\mathcal{Y}_1^{[3,1,3]}$, in the sense that $\langle\mathcal{Y}_2^{[3,1,3]}\bar{\mathcal{Y}}_1^{[3,1,3]}\rangle = 0$.

JHEP11(2001)046

$$\begin{aligned}
\mathcal{O}_6^{[2,3,2]} &\equiv -8\,\mathrm{tr}\, z_1 z_2 \mathrm{tr}\, z_1 z_2 \mathrm{tr}\, z_1 z_1 z_1 - 6\,\mathrm{tr}\, z_1 z_1 \mathrm{tr}\, z_1 z_2 \mathrm{tr}\, z_1 z_1 z_2 + \\
&\quad + \mathrm{tr}\, z_1 z_1 \left(11\,\mathrm{tr}\, z_2 z_2\ \mathrm{tr}\, z_1 z_1 z_1 + 3\,\mathrm{tr}\, z_1 z_1\ \mathrm{tr}\, z_1 z_2 z_2\right) \qquad (7.22) \\
\mathcal{O}_7^{[2,3,2]} &\equiv 7\,\mathrm{tr}\, z_1 z_2 \mathrm{tr}\, z_1 z_2 \mathrm{tr}\, z_1 z_1 z_1 - 6\,\mathrm{tr}\, z_1 z_1 \mathrm{tr}\, z_1 z_2 \mathrm{tr}\, z_1 z_1 z_2 + \\
&\quad + \mathrm{tr}\, z_1 z_1 \left(-4\,\mathrm{tr}\, z_2 z_2\ \mathrm{tr}\, z_1 z_1 z_1 + 3\,\mathrm{tr}\, z_1 z_1\ \mathrm{tr}\, z_1 z_2 z_2\right). \qquad (7.23)
\end{aligned}$$

More explicitly, the operators listed in (7.17-7.23) are constructed as

$$7 = 7: \quad \mathcal{O}_1^{[2,3,2]} \sim \left(\ydiagram{5,2} \right); \qquad (7.24)$$

$$7 = 5+2: \quad \mathcal{O}_2^{[2,3,2]} \sim \left(\begin{array}{l} \ydiagram{5} \\ \ydiagram{2} \end{array} \right), \quad \mathcal{O}_3^{[2,3,2]} \sim \left(\ydiagram{3,2}\ \ydiagram{2} \right); \qquad (7.25)$$

$$7 = 4+3: \quad \mathcal{O}_4^{[2,3,2]} \sim \left(\ydiagram{1,2}\ \ydiagram{4} \right), \quad \mathcal{O}_5^{[2,3,2]} \sim \left(\ydiagram{2,2}\ \ydiagram{3} \right); \qquad (7.26)$$

$$7 = 3+2+2: \quad \mathcal{O}_6^{[2,3,2]} \sim \left(\begin{array}{l} \ydiagram{2}\ \ydiagram{3} \\ \ydiagram{2} \end{array} \right), \quad \mathcal{O}_7^{[2,3,2]} \sim \left(\ydiagram{1,1}\ \ydiagram{1,1}\ \ydiagram{3} \right) \qquad (7.27)$$

where each continuous group of boxes stands for a single trace. Other partitions give rise to operators which are linear combinations of the ones shown in (7.24)–(7.27). Now it matters not only how we partition the string of seven letters, but exactly which letters we put in the groups. For example, $\mathcal{O}_{4,5}^{[2,3,2]}$ correspond to the same partition 7=4+3, but it is important which z_i appear in which trace before we apply the Young operator.

Again, the $\mathcal{O}_i^{[2,3,2]}$ have nonzero two point functions with each other, none are pure, and it is impossible to guess the primaries which the descendants come from. Matrices $\langle \mathcal{O}_i^{[2,3,2]} \bar{\mathcal{O}}_j^{[2,3,2]} \rangle$ of two point functions are listed in appendix C.2. The operators we predict to be $\frac{1}{4}$-BPS are: a combination of double and single trace operators only

$$\mathcal{Y}_1^{[2,3,2]} = -\frac{10N}{N^2-7}\,\mathcal{O}_1^{[2,3,2]} + \mathcal{O}_2^{[2,3,2]} + \frac{2}{N^2-7}\left(\mathcal{O}_3^{[2,3,2]} + \mathcal{O}_4^{[2,3,2]} + \mathcal{O}_5^{[2,3,2]}\right) \qquad (7.28)$$

and two linear combinations involving all types of operators

$$\begin{aligned}
\mathcal{Y}_2^{[2,3,2]} &= -20\,\mathcal{O}_1^{[2,3,2]} + \frac{2\,(N^2+2)}{N}\,\mathcal{O}_2^{[2,3,2]} - \frac{2}{N}\left(\mathcal{O}_3^{[2,3,2]} + \mathcal{O}_4^{[2,3,2]}\right) + \mathcal{O}_6^{[2,3,2]} \\
\mathcal{Y}_3^{[2,3,2]} &= 10\,\mathcal{O}_1^{[2,3,2]} - \frac{(N^2-4)}{N}\,\mathcal{O}_2^{[2,3,2]} - \frac{2}{N}\left(\mathcal{O}_3^{[2,3,2]} + \mathcal{O}_4^{[2,3,2]}\right) + \mathcal{O}_7^{[2,3,2]} \qquad (7.29)
\end{aligned}$$

($\mathcal{Y}_{1,2,3}^{[2,3,2]}$ are not orthogonal. Although orthonormal linear combinations are easy to find, they look rather messy and we don't list them here.) Again we emphasize that these expressions are exact in N and are not large N approximations.

The remaining four pure operators involve mixtures of all $\mathcal{O}_{1,\ldots,7}^{[2,3,2]}$. The coefficients are again irrational functions of N (unlike for $\mathcal{Y}_{1,2,3}^{[2,3,2]}$), and so are the eigenvalues. In the large N limit, one of them has a g^2 correction to its scaling dimension which is suppressed by N^{-2} — another operator becomes $\frac{1}{4}$-BPS in the large N limit.

$[p,q,p]$	$\mathcal{O}_i^{[p,q,p]}$	$\mathcal{Y}_j^{[p,q,p]}$
$[2,0,2]$	$\mathcal{O}_1^{[2,0,2]} \sim \left(\ydiagram{2,2}\right)$ $\mathcal{O}_2^{[2,0,2]} \sim \left(\begin{array}{l}\ydiagram{2}\\\ydiagram{2}\end{array}\right)$	$\mathcal{Y}_1^{[2,0,2]} = -\frac{4}{N}\mathcal{O}_1^{[2,0,2]} + \mathcal{O}_2^{[2,0,2]}$
$[2,1,2]$	$\mathcal{O}_1^{[2,1,2]} \sim \left(\ydiagram{3,2}\right)$ $\mathcal{O}_2^{[2,1,2]} \sim \left(\begin{array}{l}\ydiagram{3}\\\ydiagram{2}\end{array}\right)$	$\mathcal{Y}_1^{[2,1,2]} = -\frac{6}{N}\mathcal{O}_1^{[2,1,2]} + \mathcal{O}_2^{[2,1,2]}$
$[2,2,2]$	$\mathcal{O}_1^{[2,2,2]} \sim \left(\ydiagram{4,2}\right)$ $\mathcal{O}_2^{[2,2,2]} \sim \left(\begin{array}{l}\ydiagram{4}\\\ydiagram{2}\end{array}\right)$ $\mathcal{O}_3^{[2,2,2]} \sim \left(\ydiagram{2,1}\ \ydiagram{2,1}\right)$ $\mathcal{O}_4^{[2,2,2]} \sim \left(\ydiagram{2,2}\ \ydiagram{2}\right)$ $\mathcal{O}_5^{[2,2,2]} \sim \left(\begin{array}{l}\ydiagram{2}\\\ydiagram{2}\end{array}\ \ydiagram{2}\right)$	$\mathcal{Y}_1^{[2,2,2]} = -\frac{8N}{N^2-4}\mathcal{O}_1^{[2,2,2]} + \mathcal{O}_2^{[2,2,2]} + \frac{8}{3(N^2-4)}\left(2\mathcal{O}_3^{[2,2,2]} + \mathcal{O}_4^{[2,2,2]}\right)$ $\mathcal{Y}_2^{[2,2,2]} = -\frac{8}{N}\mathcal{O}_1^{[2,2,2]} + \mathcal{O}_2^{[2,2,2]} + \frac{4}{3N}\mathcal{O}_5^{[2,2,2]}$
$[3,1,3]$	$\mathcal{O}_1^{[3,1,3]} \sim \left(\ydiagram{4,3}\right)$ $\mathcal{O}_2^{[3,1,3]} \sim \left(\begin{array}{l}\ydiagram{4}\\\ydiagram{3}\end{array}\right)$ $\mathcal{O}_3^{[3,1,3]} \sim \left(\ydiagram{2,3}\ \ydiagram{2}\right)$ $\mathcal{O}_4^{[3,1,3]} \sim \left(\begin{array}{l}\ydiagram{2}\\\ydiagram{3}\end{array}\ \ydiagram{2}\right)$	$\mathcal{Y}_1^{[3,1,3]} = -\frac{12N}{N^2-2}\mathcal{O}_1^{[3,1,3]} + \mathcal{O}_2^{[3,1,3]} - \frac{5}{N^2-2}\mathcal{O}_3^{[3,1,3]}$ $\mathcal{Y}_2^{[3,1,3]} = -\frac{24}{N}\mathcal{O}_1^{[3,1,3]} + \mathcal{O}_2^{[3,1,3]} + \frac{1}{2N}\mathcal{O}_4^{[3,1,3]}$
$[2,3,2]$	$\mathcal{O}_1^{[2,3,2]} \sim \left(\ydiagram{5,2}\right)$ $\mathcal{O}_2^{[2,3,2]} \sim \left(\begin{array}{l}\ydiagram{5}\\\ydiagram{2}\end{array}\right)$ $\mathcal{O}_3^{[2,3,2]} \sim \left(\ydiagram{3,2}\ \ydiagram{2}\right)$ $\mathcal{O}_4^{[2,3,2]} \sim \left(\ydiagram{1,2}\ \ydiagram{4}\right)$ $\mathcal{O}_5^{[2,3,2]} \sim \left(\ydiagram{2,2}\ \ydiagram{3}\right)$ $\mathcal{O}_6^{[2,3,2]} \sim \left(\begin{array}{l}\ydiagram{2}\\\ydiagram{2}\end{array}\ \ydiagram{3}\right)$ $\mathcal{O}_7^{[2,3,2]} \sim \left(\ydiagram{1,1}\ \ydiagram{1,1}\ \ydiagram{3}\right)$	$\mathcal{Y}_1^{[2,3,2]} = -\frac{10N}{N^2-7}\mathcal{O}_1^{[2,3,2]} + \mathcal{O}_2^{[2,3,2]} + \frac{2}{N^2-7}\left(\mathcal{O}_3^{[2,3,2]} + \mathcal{O}_4^{[2,3,2]} + \mathcal{O}_5^{[2,3,2]}\right)$ $\mathcal{Y}_2^{[2,3,2]} = -20\mathcal{O}_1^{[2,3,2]} + \frac{2N^2+4}{N}\mathcal{O}_2^{[2,3,2]} - \frac{2}{N}\left(\mathcal{O}_3^{[2,3,2]} + \mathcal{O}_4^{[2,3,2]}\right) + \mathcal{O}_6^{[2,3,2]}$ $\mathcal{Y}_3^{[2,3,2]} = 10\mathcal{O}_1^{[2,3,2]} - \frac{N^2-4}{N}\mathcal{O}_2^{[2,3,2]} - \frac{2}{N}\left(\mathcal{O}_3^{[2,3,2]} + \mathcal{O}_4^{[2,3,2]}\right) + \mathcal{O}_7^{[2,3,2]}$

Table 1: Gauge invariant, local, polynomial, scalar composite operators in the $[p,q,p]$ representations of SU(4), with $2p+q \leq 7$. Each continuous string of boxes in the $\mathcal{O}_i^{[p,q,p]}$ corresponds to a single trace. The $\mathcal{Y}_i^{[p,q,p]}$ listed are the $\frac{1}{4}$-BPS chiral primaries; other pure operators are not shown.

8. Summary of $\Delta \leq 7$ results

Having carried out these explicit calculations, let us bring together the results for $[p, q, p]$ highest weight, gauge invariant, local, polynomial scalar composite operators we have discussed so far. Table 1 lists the representations $[p, q, p]$ with $2p + q \leq 7$; the operators $\mathcal{O}_i^{[p,q,p]}$ constructed by taking traces in various combinations; and the resulting pure operators $\mathcal{Y}_j^{[p,q,p]}$. We omitted the $\mathcal{Y}_j^{[p,q,p]}$ with corrected scaling dimension, and listed only the $\frac{1}{4}$-BPS chiral primaries. The notation and definitions are spelled out in sections 5 and 7.

9. Large N analysis

As we have seen, computations get more and more cumbersome as one tries to find $\frac{1}{4}$-BPS operators for bigger representations of the color group; even the number of operators one has to consider is a nontrivial function of the representation. Symmetry factors multiplying the Feynman graphs show no immediate pattern, and most of the results presented in sections 5.1, 5.2, and 7 had to be calculated using *Mathematica*.[24]

The next best thing we can do is consider the large N limit. Specifically, we shall concentrate on the leading behavior as $N \to \infty$, plus the first $1/N$ correction.

9.1 Operators $\mathcal{O}_{[p,q,p]}$ and $\mathcal{K}_{[p,q,p]}$

Let us take another look at the results of section 7, where we managed to perform $\mathcal{O}(g^2)$ analysis exactly in N rather than in the large N approximation. In all cases considered so far, there is a special $\frac{1}{4}$-BPS chiral primary $\mathcal{Y}_1^{[2,3,2]}$ (equations (7.6), (7.14), and (7.28)), which is made of only the double trace and single trace operators. At large N, this operator is a combination of only a particular double trace operator, and the single trace operator, whose contribution is $1/N$ suppressed. The goal of section 9 is to show that this is in fact what happens for general $[p, q, p]$ representations. Here, we begin by defining these operators.

Recall that the SO(6) Young tableau for the $[p, q, p]$ of SU(4) consists of two rows (one of length $p + q$, and the other of length p). Among the possible partitions of the highest weight tableau, there are two special ones

$$\mathcal{O}_{[p,q,p]} \sim \left(\begin{array}{l} \boxed{1}\boxed{\cdots}\boxed{1}\;\boxed{1}\boxed{\cdots}\boxed{1} \\ \underbrace{\boxed{2}\boxed{\cdots}\boxed{2}}_{p}\quad\; q \end{array} \right), \quad \mathcal{K}_{[p,q,p]} \sim \left(\begin{array}{l} \boxed{1}\boxed{\cdots}\boxed{1}\boxed{1}\boxed{\cdots}\boxed{1} \\ \underbrace{\boxed{2}\boxed{\cdots}\boxed{2}}_{p}\quad q \end{array} \right), \tag{9.1}$$

[24]The calculations took from 0.003 hours for the [2,0,2] representation to 23 hours for [3,1,3], per single $\mathcal{O}(g^2)$ two point function. We used a Sparc 10 with 2048 M memory and 440 MHz speed. Born level calculations were considerably (about 20 times) faster.

JHEP11(2001)046

where each continuous group of boxes stands for a single trace, as before. Explicitly, the corresponding operators are

$$\mathcal{O}_{[p,q,p]} = \sum_{k=0}^{p} \frac{(-1)^k p!}{k!(p-k)!}\ \mathrm{tr}\ \left({z_1}^{p+q-k}{z_2}^k\right)_s\ \mathrm{tr}\ \left({z_1}^k{z_2}^{p-k}\right)_s \tag{9.2}$$

$$\mathcal{K}_{[p,q,p]} = \sum_{k=0}^{p} \frac{(-1)^k p!}{k!(p-k)!}\ \mathrm{tr}\ \left[\left({z_1}^{p+q-k}{z_2}^k\right)_s \left({z_1}^k{z_2}^{p-k}\right)_s\right] \tag{9.3}$$

(after projecting onto $\mathrm{SU}(4) \to \mathrm{SU}(3)\times\mathrm{U}(1)$ and keeping only the highest U(1)-charge pieces, as discussed in sections 2 and 5.1). Made of only z_1 and z_2, both types of operators are annihilated by four out of the sixteen Poincaré supersymmetry generators: using the SUSY transformations spelled out in appendix A, we find $\bar{Q}_{\bar\zeta} z_j = 0$, $Q_{\zeta_3} z_j = -\sqrt{2}(\lambda\zeta_3)\delta_{3j}$, so

$$\bar{Q}_{\bar\zeta}\mathcal{O}_{[p,q,p]} = Q_{\zeta_3}\mathcal{O}_{[p,q,p]} = \bar{Q}_{\bar\zeta}\mathcal{K}_{[p,q,p]} = Q_{\zeta_3}\mathcal{K}_{[p,q,p]} = 0\,. \tag{9.4}$$

It is clear why $\mathcal{K}_{[p,q,p]}$ is special: it is the only single trace $[p,q,p]$ operator which can be constructed out of these fields. On the other hand, $\mathcal{O}_{[p,q,p]}$ is "the most natural" double trace composite operator in this representation. We also recognize it as the free theory chiral primary from the classification of [2].

As we have seen in the previous sections, neither the single trace $\mathcal{K}_{[p,q,p]}$ nor the double trace $\mathcal{O}_{[p,q,p]}$ are eigenstates of the dilation operator, for general N. Below we calculate correlators $\langle\mathcal{O}\bar{\mathcal{O}}\rangle$, $\langle\mathcal{O}\bar{\mathcal{K}}\rangle$, $\langle\mathcal{K}\bar{\mathcal{O}}\rangle$, and $\langle\mathcal{K}\bar{\mathcal{K}}\rangle$, in the large N limit, and determine the pure operators and their scaling dimension in this approximation.

9.2 General correlators $\langle\mathcal{O}_{[p,q,p]}\bar{\mathcal{O}}_{[p,q,p]}\rangle$ to order g^2

Let us first consider the $\langle\mathcal{O}_{[p,q,p]}(x)\bar{\mathcal{O}}_{[p,q,p]}(y)\rangle$ correlators. The free contribution is just a power of the free scalar propagator $G(x,y) = [4\pi(x-y)^2]^{-1}$, times a combinatorial factor:

$$\begin{aligned}\langle\mathcal{O}_{[p,q,p]}(x)\bar{\mathcal{O}}_{[p,q,p]}(y)\rangle|_{\mathrm{free}} &= \sum_{k,l=0}^{p} \frac{(-1)^k p!}{k!(p-k)!}\ \frac{(-1)^l p!}{l!(p-l)!}\ (\mathcal{R}^{p+q,p}_{k,l})|_{\mathrm{free}} \\ &= [G(x,y)]^{(2p+q)} \sum_{k,l=0}^{p} \frac{(-1)^k p!}{k!(p-k)!}\ \frac{(-1)^l p!}{l!(p-l)!}\ \mathcal{F}^{p+q,p}_{k,l}\,,\end{aligned} \tag{9.5}$$

where

$$\begin{aligned}\mathcal{R}^{p+q,p}_{k,l} = \langle &\left[\mathrm{str}\,(z_1)^{(p+q-k)}(z_2)^k\right](x)\left[\mathrm{str}\,(z_1)^k(z_2)^{(p-k)}\right](x) \\ &\left[\mathrm{str}\,(z_1)^{(p+q-l)}(z_2)^l\right](y)\left[\mathrm{str}\,(z_1)^l(z_2)^{(p-l)}\right](y)\rangle\end{aligned} \tag{9.6}$$

and

$$\begin{aligned}\mathcal{F}^{p+q,p}_{k,l} = \sum_{\sigma,\rho} &\left[\mathrm{str}\,t^{a_{k+1}}\dots t^{a_{p+q}}t^{b_1}\dots t^{b_k}\right]\left[\mathrm{str}\,t^{a_1}\dots t^{a_k}t^{b_{k+1}}\dots t^{b_p}\right]\times \\ &\times\left[\mathrm{str}\,t^{a_{\sigma(l+1)}}\dots t^{a_{\sigma(p+q)}}t^{b_{\rho(1)}}\dots t^{b_{\rho(l)}}\right]\left[\mathrm{str}\,t^{a_{\sigma(1)}}\dots t^{a_{\sigma(l)}}t^{b_{\rho(l+1)}}\dots t^{b_{\rho(p)}}\right]\end{aligned} \tag{9.7}$$

(σ and ρ sample over groups of permutations S_{p+q} and S_p on $p+q$ and p letters, respectively).

Like in the $\frac{1}{2}$-BPS case, the leading contribution[25] to $\mathcal{F}^{p+q,p}_{k,l} \sim (N/2)^{(2p+q)}$ comes from terms in which generators appear in reverse order for z-s and $\bar{z}$-s. To estimate the large N behavior we can use equation (B.22) to "merge traces," $(\mathrm{tr}\, t^{d_1} \dots t^{d_s} t^c)(\mathrm{tr}\, t^c t^{d_s} \dots t^{d_1}) \sim \frac{1}{2}\mathrm{tr}\, t^{d_1} \dots t^{d_s} t^{d_s} \dots t^{d_1} \sim (N/2)^{s+1}$. In order to find the numerical factor out front (which does not scale with N but depends on p and q), we should determine exactly which terms have this structure.

The generators can appear in opposite order in two pairs of traces in (9.7) under the following circumstances. First, it can happen when $k = l$ and the traces are merged as 1 with 3 and 2 with 4. The factors which arise are: $[1/p!]^2$ from symmetrizations in the 2-nd and 4-th traces; $[1/(p+q)!]^2$ from symmetrizations in the 2-nd and 4-th traces; $p!$ because for any ordering in the 1-st trace there is an identical one in the 3-d trace; $(p+q)!$ for the same reason for the 2-nd and 4-th trace; $k!(p-k)!$ since any permutation of just t^a-s or just t^b-s in the 1-st trace can be "undone" by σ-s and ρ-s in the 3-d trace; and similarly $k!(p+q-k)!$ for the 2-nd and 4-th trace; $p(p+q)$ because of trace cyclicity. There is also an overall factor from the definition (9.2). Second, if $q = 0$, we can merge traces the other way: 1 with 4 and 2 with 3; in this case $k = p - l$ and all other factors are the same. Thus, the leading contributions[26] add up to

$$
\begin{aligned}
\langle \mathcal{O}_{[p,q,p]}(x) \bar{\mathcal{O}}_{[p,q,p]}(y) \rangle|_{\mathrm{free}} \sim & \left(\frac{1}{2} N G(x,y)\right)^{(2p+q)} \times \\
& \times \sum_{k=0}^{p} (-)^k \left[\frac{p!}{k!(p-k)!}\right]^2 \frac{k!(p-k)!k!(p+q-k)!}{(p-1)!(p+q-1)!} \times \\
& \times \left[(-)^k + \delta_{q,0}(-)^{p-k}\right] \qquad (9.8) \\
= & \left(\frac{1}{2} N G(x,y)\right)^{(2p+q)} \left[1 + \delta_{q,0}(-)^p\right] \frac{p(p+q)(p+q+1)}{(q+1)} .
\end{aligned}
$$

The reproduces the leading order correlators in the low dimensional cases considered in sections 5.1, 5.2, and 7. Also note that if $q = 0$ and p is odd, both operators $\mathcal{O}_{[p,q,p]}$ and $\mathcal{K}_{[p,q,p]}$ vanish identically, in agreement with (9.8).

Now consider the corrections to this result. Diagrams contributing to two point functions of scalar composite operators at $\mathcal{O}(g^2)$ fall into two categories, see figure 2. On the one hand, there are Feynman graphs involving a gauge boson exchange (these are proportional to A or B). On the other hand, we also have gauge independent ones (proportional to $\tilde{B}$) coming entirely from the $zz\bar{z}\bar{z}$-vertex. These two types of corrections have different combinatorial (index) structure, and we shall handle them separately.

[25]See appendix B for useful SU(N) identities.

[26]The error we are committing is of order $\mathcal{O}(N^{-2})$.

JHEP11(2001)046

9.2.1 Gauge dependent contributions: combinatorial argument

In section 6, we argued that two point functions of gauge-invariant operators can not contain pieces proportional to the gauge dependent functions A and B. Here we show this explicitly for operators $\mathcal{O}_{[p,q,p]}$ and $\mathcal{K}_{[p,q,p]}$. This is the only part of section 9 which is exact in N, and is not just a large N approximation.

The simplest order g^2 contribution to $\langle \mathcal{O}_{[p,q,p]} \bar{\mathcal{O}}_{[p,q,p]} \rangle$ comes from corrections to the scalar propagator (diagrams of type (a) and (b) in figure 2). It has the same index structure as the free field result, and so is the same up to a factor $(p+q)NA$ for (a)- and pNA for (b)-type diagrams. These factors simply count the number of z_1-s and z_2-s.

Next consider the other diagrams where the correction comes from blocks with the same flavor in the four legs, ones of type (d) and (e). Each term in the k, l sum in (9.5) receives corrections of the form

$$\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)(-1)(2)B\sum_{\sigma,\rho}\sum_{i\neq j=1}^{p+q}\left[\operatorname{tr} t^{b_{k+1}}\dots t^{b_p}t^{a_1}\dots t^{a_k}\right]\left[\operatorname{tr} t^{a_{k+1}}\dots t^{a_{p+q}}t^{b_1}\dots t^{b_k}\right]\times$$
$$\times\left[t^{b_{\rho(l+1)}}\dots t^{b_{\rho(p)}}t^{a_{\sigma(1)}}\dots[t^{a_{\sigma(i)}},t^c]\dots[t^{a_{\sigma(j)}},t^c]\dots t^{a_{\sigma(p+q)}}t^{b_{\rho(1)}}\dots t^{b_{\rho(l)}}\right], \tag{9.9}$$

where all traces have to be symmetrized, and the second line also contains two traces.[27] The prefactor multiplying the sum comes about in the following manner: a factor of $(\frac{1}{2})$, since the sum is on $i\neq j$ rather than on $i<j$; similarly the other $(\frac{1}{2})$ arises because using $\{\sigma(i),\sigma(j)\}$ and $\{\sigma(j),\sigma(i)\}$ counts the same pair twice; a factor of 2 has to be taken into account as the two cross-symmetric pairs give the same contribution; and finally (-1) is there from two factors of i which are needed to convert f-s to commutators.

The structure in the sum (9.9) consists of four kinds of terms. The two commutators can be both in the third trace, or both in the fourth trace, or one in either trace. When both commutators are in the same trace, we can play the same game as for $\frac{1}{2}$-BPS operators. Fix i and do the sum on j first; this assembles $(\dots[t^{a_{\sigma(1)}},t^c]\dots)+\dots+(\dots[t^{a_{\sigma(l)}},t^c]\dots) = (\dots[t^{a_{\sigma(1)}}\dots t^{a_{\sigma(l)}},t^c]\dots)$ for example. Then, use trace cyclicity in the form $\operatorname{tr}[A,B]C=\operatorname{tr}A[B,C]$ to move one of the traces over to the other commutator and t^{b_ρ}-s; here we pick up a minus sign which cancels the (-1) in (9.9). As $[[t^a,t^c],t^c]=Nt^a$, the first bit is easy — just like in the $\frac{1}{2}$-BPS case, it is proportional to $(+\frac{1}{2}BN)$ times the free-field combinatorial factor; when we do the sum on i we also get a factor of $(p+q)$ here. As $B+2A=0$ (by $\mathcal{N}$=4 SUSY), this part cancels the diagrams of type (a). The leftovers, together with the terms with commutators in different traces, add up to

$$\frac{1}{2}B\left[\operatorname{tr} t^{b_{k+1}}\dots t^{b_p}t^{a_1}\dots t^{a_k}\right]\left[\operatorname{tr} t^{a_{k+1}}\dots t^{a_{p+q}}t^{b_1}\dots t^{b_k}\right]\times$$

[27] We will omit the $[G(x,y)]^{(2p+q)}$ factor which is common to all diagrams.

$$
\begin{aligned}
&\times \sum_{\sigma,\rho} \left[\mathrm{tr}\, t^{b_{\rho(l+1)}} \dots t^{b_{\rho(p)}} t^{a_{\sigma(1)}} \dots t^{a_{\sigma(l)}}\right] \left[\mathrm{tr}\, \left[t^{a_{\sigma(l+1)}} \dots t^{a_{\sigma(p+q)}}, t^c\right] \left[t^{b_{\rho(1)}} \dots t^{b_{\rho(l)}}, t^c\right]\right] + \\
&\qquad + \left[\mathrm{tr}\, \left[t^{b_{\rho(l+1)}} \dots t^{b_{\rho(p)}}, t^c\right] \left[t^{a_{\sigma(1)}} \dots t^{a_{\sigma(l)}}, t^c\right]\right] \left[\mathrm{tr}\, t^{a_{\sigma(l+1)}} \dots t^{a_{\sigma(p+q)}} t^{b_{\rho(1)}} \dots t^{b_{\rho(l)}}\right] - \\
&\qquad - 2 \left[\mathrm{tr}\, t^{b_{\rho(l+1)}} \dots t^{b_{\rho(p)}} \left[t^{a_{\sigma(1)}} \dots t^{a_{\sigma(l)}}, t^c\right]\right] \left[\mathrm{tr}\, \left[t^{a_{\sigma(l+1)}} \dots t^{a_{\sigma(p+q)}}, t^c\right] t^{b_{\rho(1)}} \dots t^{b_{\rho(l)}}\right]
\end{aligned}
\tag{9.10}
$$

Similarly, there are diagrams of type (e) in figure 2, where all of the flavors are "2" in all four legs. Here, we have a term proportional to the free-field result (the only difference being an overall factor of p rather than $p+q$), which cancels contributions from diagrams of type (b). The leftover term is the same as what we have just computed. This removes the factor of $\frac{1}{2}$ from (9.10).

Finally consider the diagrams of type (f). Now we have both flavors "1" and "2" in the four-scalar blocks, while the index structure is the same as that of (d) and (e). The discussion goes through as above, but with a few small modifications. First, the prefactor is just $(-1)B$ as now the indices i and j run over different flavors (so there is no "$i \neq j$ overcounting," no "$\{\sigma(i), \sigma(j)\}$ overcounting," and no factor of 2 from crossing symmetry). Second, we do not pick up a minus sign when transferring commutators under the traces (before, both commutators with t^c were on, say, t^a-s, whereas now one is on t^a and one on t^b). Therefore, the result of adding the diagrams of type (f) is to precisely cancel the whole leftover structure of (twice that given in) equation (9.10).

Thus we have explicitly reproduced the general result of section 6, but with a lot more work: A and B contributions to two-point functions of arbitrary gauge-invariant scalar composite operators combine as $2A + B$, which vanishes by $\mathcal{N}$=4 supersymmetry.

9.2.2 Contributions proportional to $\tilde{B}$

So far we have established that adding all gauge dependent Feynman graphs, i.e. diagrams of types (a), (b), (d), (e), and (f) shown in figure 2, gives a vanishing $\mathcal{O}(g^2)$ contribution to the two-point functions (9.5), once we impose $\mathcal{N}$=4 SUSY. Just as in the cases of low dimensional operators considered in sections 5 and 7, the $\mathcal{O}(g^2)$ corrections to the two-point function of $[p,q,p]$ operators come exclusively from diagrams of type (c), and are proportional to $\tilde{B}$. To find the combinatorial factor multiplying $\tilde{B}$, we need to perform a calculation similar to the one in section 9.2.1.

Here, "contracted with f-s" are z-s with z-s and $\bar{z}$-s with $\bar{z}$-s (unlike in diagrams of types (d), (e) and (f), where it was one z and one $\bar{z}$), and it is more convenient to label the generators slightly differently. For example, the free-field result (9.7) can be rewritten as

$$
\begin{aligned}
\mathcal{F}^{p+q,p}_{k,l} = &\sum_{\sigma,\rho} \left[\mathrm{str}\, t^{a_{\sigma(k+1)}} \dots t^{a_{\sigma(p+q)}} t^{b_1} \dots t^{b_k}\right] \left[\mathrm{str}\, t^{a_{\sigma(1)}} \dots t^{a_{\sigma(k)}} t^{b_{k+1}} \dots t^{b_p}\right] \times \\
&\times \left[\mathrm{str}\, t^{a_{l+1}} \dots t^{a_{p+q}} t^{b_{\rho(1)}} \dots t^{b_{\rho(l)}}\right] \left[\mathrm{str}\, t^{a_1} \dots t^{a_l} t^{b_{\rho(l+1)}} \dots t^{b_{\rho(p)}}\right].
\end{aligned}
\tag{9.11}
$$

For the Born level combinatorial factor it makes no difference, but in calculating the order g^2 diagrams of type (c) we "f-contract" i-th z_1 with $\rho(j)$-th z_2, and $\sigma(i)$-th $\bar{z}_1$ with j-th $\bar{z}_2$. This will exhaust all pairs without overcounting (because i and j are again on different flavors), so the prefactor will be just $(-1)\tilde{B}(x,y)$. Apart from this prefactor (and a factor of $[G(x,y)]^{(2p+q)}$), the (c)-type correction reads (sums on σ and ρ implied)

$$
\begin{aligned}
&\sum \left[\mathrm{tr}\, t^{a_{\sigma(k+1)}} \dots t^{a_{\sigma(p+q)}} t^{b_1} \dots t^{b_{j-1}} [t^{a_i}, t^c] t^{b_{j+1}} \dots t^{b_k}\right] \left[\mathrm{tr}\, t^{a_{\sigma(1)}} \dots t^{a_{\sigma(k)}} t^{b_{k+1}} \dots t^{b_p}\right] \times \\
&\times \left[\mathrm{tr}\, t^{a_{l+1}} \dots t^{a_{p+q}} t^{b_{\rho(1)}} \dots t^{b_{\rho(l)}}\right] \left[\mathrm{tr}\, t^{a_1} \dots t^{a_{i-1}} [t^{b_j}, t^c] t^{a_{i+1}} \dots t^{a_l} t^{b_{\rho(l+1)}} \dots t^{b_{\rho(p)}}\right] + \\
&+ \sum \left[\mathrm{tr}\, t^{a_{\sigma(k+1)}} \dots t^{a_{\sigma(p+q)}} t^{b_1} \dots t^{b_{j-1}} [t^{a_i}, t^c] t^{b_{j+1}} \dots t^{b_k}\right] \left[\mathrm{tr}\, t^{a_{\sigma(1)}} \dots t^{a_{\sigma(k)}} t^{b_{k+1}} \dots t^{b_p}\right] \times \\
&\times \left[\mathrm{tr}\, t^{a_{l+1}} \dots t^{a_{i-1}} [t^{b_j}, t^c] t^{a_{i+1}} \dots t^{a_{p+q}} t^{b_{\rho(1)}} \dots t^{b_{\rho(l)}}\right] \left[\mathrm{tr}\, t^{a_1} \dots t^{a_l} t^{b_{\rho(l+1)}} \dots t^{b_{\rho(p)}}\right] + \\
&+ \sum \left[\mathrm{tr}\, t^{a_{\sigma(k+1)}} \dots t^{a_{\sigma(p+q)}} t^{b_1} \dots t^{b_k}\right] \left[\mathrm{tr}\, t^{a_{\sigma(1)}} \dots t^{a_{\sigma(k)}} t^{b_{k+1}} \dots t^{b_{j-1}} [t^{a_i}, t^c] t^{b_{j+1}} \dots t^{b_p}\right] \times \\
&\times \left[\mathrm{tr}\, t^{a_{l+1}} \dots t^{a_{p+q}} t^{b_{\rho(1)}} \dots t^{b_{\rho(l)}}\right] \left[\mathrm{tr}\, t^{a_1} \dots t^{a_{i-1}} [t^{b_j}, t^c] t^{a_{i+1}} \dots t^{a_l} t^{b_{\rho(l+1)}} \dots t^{b_{\rho(p)}}\right] + \\
&+ \sum \left[\mathrm{tr}\, t^{a_{\sigma(k+1)}} \dots t^{a_{\sigma(p+q)}} t^{b_1} \dots t^{b_k}\right] \left[\mathrm{tr}\, t^{a_{\sigma(1)}} \dots t^{a_{\sigma(k)}} t^{b_{k+1}} \dots t^{b_{j-1}} [t^{a_i}, t^c] t^{b_{j+1}} \dots t^{b_p}\right] \times \\
&\times \left[\mathrm{tr}\, t^{a_{l+1}} \dots t^{a_{i-1}} [t^{b_j}, t^c] t^{a_{i+1}} \dots t^{a_{p+q}} t^{b_{\rho(1)}} \dots t^{b_{\rho(l)}}\right] \left[\mathrm{tr}\, t^{a_1} \dots t^{a_l} t^{b_{\rho(l+1)}} \dots t^{b_{\rho(p)}}\right],
\end{aligned}
$$

where all traces have to be symmetrized again; the sums on $\{i,j\}$ are: in the first line, from $\{1,1\}$ to $\{l,k\}$, in the second line, from $\{l+1,1\}$ to $\{p+q,k\}$, in the third line, from $\{1,k+1\}$ to $\{l,p\}$, and in the last line, from $\{l+1,k+1\}$ to $\{p+q,p\}$.

In the large N limit, such terms can scale as $\frac{1}{2}(N/2)^{(2p+q-1)}$, at best. (Because of the commutators with t^c, we have to merge traces three times: they don't eat each other up in pairs as they did before). After including the factor of $\tilde{B}$, together with (9.8) this means that

$$
\begin{aligned}
\langle \mathcal{O}_{[p,q,p]}(x) \bar{\mathcal{O}}_{[p,q,p]}(y) \rangle = {} & \left(\frac{N}{2} G(x,y)\right)^{(2p+q)} \times \\
& \times \left([1 + \delta_{q,0}(-)^p] \, \frac{p(p+q)(p+q+1)}{(q+1)} + \tilde{B}(x,y) N \times \mathcal{O}(N^{-2}) \right)
\end{aligned} \tag{9.12}
$$

to order g^2. We might be tempted to stop here. By observing that since to working precision, the two point function of $\mathcal{O}_{[p,q,p]}$ with itself does not get $\mathcal{O}(g^2)$ corrections, we could try to conclude it is chiral, and in particular has a protected $\Delta = 2p+q$. However, as the explicit examples of sections 5 and 7 show, $\mathcal{O}_{[p,q,p]}$ may not be a pure operator, in which case it doesn't make sense to talk about its scaling dimension.

9.3 General correlators $\langle \mathcal{K}_{[p,q,p]} \bar{\mathcal{O}}_{[p,q,p]} \rangle$

The analysis exactly parallels that of the previous section. Again,

$$
\langle \mathcal{K}_{[p,q,p]}(x) \bar{\mathcal{O}}_{[p,q,p]}(y) \rangle = \sum_{k,l=0}^{p} \frac{(-1)^k p!}{k!(p-k)!} \, \frac{(-1)^l p!}{l!(p-l)!} \, \mathcal{P}_{k,l}^{p+q,p} \tag{9.13}
$$

JHEP11(2001)046

with

$$\mathcal{P}^{p+q,p}_{k,l} = \langle \mathrm{tr}\ \left[\left((z_1)^{(p+q-l)}(z_2)^l\right)_s \left((z_1)^l(z_2)^{(p-l)}\right)_s\right](x) \times \\ \times \left[\mathrm{str}\,(z_1)^{(p+q-k)}(z_2)^k\right](y) \left[\mathrm{str}\,(z_1)^k(z_2)^{(p-k)}\right](y)\rangle\,. \tag{9.14}$$

The leading large N contributions to the free correlators now come from terms which contain the combinatorial factor similar to

$$(\mathrm{tr}\, t^{a_1}\dots t^{a_k}t^{b_{k+1}}\dots t^{b_p}t^{a_{k+1}}\dots t^{a_{p+q}}t^{b_1}\dots t^{b_k}) \times \\ \times(\mathrm{tr}\, t^{b_p}\dots t^{b_{k+1}}t^{a_k}\dots t^{a_1})(\mathrm{tr}\, t^{b_k}\dots t^{b_1}t^{a_{p+q}}\dots t^{a_{k+1}}) \\ \sim \left(\frac{1}{2}\right)^2 N(N/2)^{(2p+q-2)} = \frac{1}{2}(N/2)^{(2p+q-1)} \tag{9.15}$$

(the two halves and -2 in the exponent are because we have to merge traces twice, and the factor of $N=\mathrm{tr}\,\mathbf{1}$ is there as usual). To get the other numerical factors we have to carefully analyze which terms in the sums and symmetrizations scale with N this way. So far, we do not need them.

As before, individual terms in the k, l sum get corrections similar to (9.9):

$$\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)(-1)(2)B\sum_{\sigma,\rho}\sum_{i\neq j=1}^{p+q} \mathrm{tr}\ \left[\left(t^{b_{k+1}}\dots t^{b_p}t^{a_1}\dots t^{a_k}\right)\left(t^{a_{k+1}}\dots t^{a_{p+q}}t^{b_1}\dots t^{b_k}\right)\right] \times \\ \times\left[t^{b_{\rho(l+1)}}\dots t^{b_{\rho(p)}}t^{a_{\sigma(1)}}\dots\left[t^{a_{\sigma(i)}},t^c\right]\dots\left[t^{a_{\sigma(j)}},t^c\right]\dots t^{a_{\sigma(p+q)}}t^{b_{\rho(1)}}\dots t^{b_{\rho(l)}}\right] \tag{9.16}$$

with proper symmetrizations (and omitted factor of $[G(x,y)]^{(2p+q)}$). The only difference is that now there are three traces (rather than four). The discussion of gauge dependent diagrams (all types other than (c), see figure 2) goes through verbatim, since we were only manipulating the second set of traces. As before, when we impose $\mathcal{N}$=4 SUSY they cancel, and order g^2 corrections to the two-point functions (9.13) are due to diagrams of type (c) only.

Diagrams of type (c) are only slightly different from those contributing to the $\langle\mathcal{O}\bar{\mathcal{O}}\rangle$ correlator; they add up (apart from the $(-1)\tilde{B}(x,y)[G(x,y)]^{(2p+q)}$ prefactor) to

$$\sum \mathrm{tr}\left[\left(t^{a_{\sigma(k+1)}}\dots t^{a_{\sigma(p+q)}}t^{b_1}\dots t^{b_{j-1}}\left[t^{a_i},t^c\right]t^{b_{j+1}}\dots t^{b_k}\right)\left(t^{a_{\sigma(1)}}\dots t^{a_{\sigma(k)}}t^{b_{k+1}}\dots t^{b_p}\right)\right] \times \\ \times\left[\mathrm{tr}\, t^{a_{l+1}}\dots t^{a_{p+q}}t^{b_{\rho(1)}}\dots t^{b_{\rho(l)}}\right]\left[\mathrm{tr}\, t^{a_1}\dots t^{a_{i-1}}\left[t^{b_j},t^c\right]t^{a_{i+1}}\dots t^{a_l}t^{b_{\rho(l+1)}}\dots t^{b_{\rho(p)}}\right] + \\ +\sum \mathrm{tr}\left[\left(t^{a_{\sigma(k+1)}}\dots t^{a_{\sigma(p+q)}}t^{b_1}\dots t^{b_{j-1}}\left[t^{a_i},t^c\right]t^{b_{j+1}}\dots t^{b_k}\right)\left(t^{a_{\sigma(1)}}\dots t^{a_{\sigma(k)}}t^{b_{k+1}}\dots t^{b_p}\right)\right] \times \\ \times\left[\mathrm{tr}\, t^{a_{l+1}}\dots t^{a_{i-1}}\left[t^{b_j},t^c\right]t^{a_{i+1}}\dots t^{a_{p+q}}t^{b_{\rho(1)}}\dots t^{b_{\rho(l)}}\right]\left[\mathrm{tr}\, t^{a_1}\dots t^{a_l}t^{b_{\rho(l+1)}}\dots t^{b_{\rho(p)}}\right] + \\ +\sum \mathrm{tr}\left[\left(t^{a_{\sigma(k+1)}}\dots t^{a_{\sigma(p+q)}}t^{b_1}\dots t^{b_k}\right)\left(t^{a_{\sigma(1)}}\dots t^{a_{\sigma(k)}}t^{b_{k+1}}\dots t^{b_{j-1}}\left[t^{a_i},t^c\right]t^{b_{j+1}}\dots t^{b_p}\right)\right] \times \\ \times\left[\mathrm{tr}\, t^{a_{l+1}}\dots t^{a_{p+q}}t^{b_{\rho(1)}}\dots t^{b_{\rho(l)}}\right]\left[\mathrm{tr}\, t^{a_1}\dots t^{a_{i-1}}\left[t^{b_j},t^c\right]t^{a_{i+1}}\dots t^{a_l}t^{b_{\rho(l+1)}}\dots t^{b_{\rho(p)}}\right] + \\ +\sum \mathrm{tr}\left[\left(t^{a_{\sigma(k+1)}}\dots t^{a_{\sigma(p+q)}}t^{b_1}\dots t^{b_k}\right)\left(t^{a_{\sigma(1)}}\dots t^{a_{\sigma(k)}}t^{b_{k+1}}\dots t^{b_{j-1}}\left[t^{a_i},t^c\right]t^{b_{j+1}}\dots t^{b_p}\right)\right] \times \\ \times\left[\mathrm{tr}\, t^{a_{l+1}}\dots t^{a_{i-1}}\left[t^{b_j},t^c\right]t^{a_{i+1}}\dots t^{a_{p+q}}t^{b_{\rho(1)}}\dots t^{b_{\rho(l)}}\right]\left[\mathrm{tr}\, t^{a_1}\dots t^{a_l}t^{b_{\rho(l+1)}}\dots t^{b_{\rho(p)}}\right] \tag{9.17}$$

with proper symmetrizations; the sums on $\{i,j\}$ are: in the first line, from $\{1,1\}$ to $\{l,k\}$, in the second line, from $\{l+1,1\}$ to $\{p+q,k\}$, in the third line, from $\{1,k+1\}$ to $\{l,p\}$, and in the last line, from $\{l+1,k+1\}$ to $\{p+q,p\}$.

We can get correlators $\langle \mathcal{O}_{[p,q,p]}(x)\bar{\mathcal{K}}_{[p,q,p]}(y)\rangle$ by just complex conjugating the sum (9.17) times the same prefactor; both the free propagator $G(x,y)$ and the function $\tilde{B}(x,y)$ are real and depend only on $(x-y)^2$, so exchanging the arguments $x \leftrightarrow y$ and conjugating doesn't change anything.

Large N dependence of (9.17) can be again estimated by merging traces. This time, we have to merge traces only twice (there are three traces total), so it scales a power of N higher than a similar $\langle \mathcal{O}(x)\bar{\mathcal{O}}(y)\rangle|_{g^2}$ correction (where traces had to be merged three times). Hence, $\langle \mathcal{O}_{[p,q,p]}(x)\bar{\mathcal{K}}_{[p,q,p]}(y)\rangle|_{g^2} \sim (N/2)^{(2p+q)}$.

9.4 General correlators $\langle \mathcal{K}_{[p,q,p]}\bar{\mathcal{K}}_{[p,q,p]}\rangle$

The analysis is similar as for $\langle \mathcal{K}_{[p,q,p]}(x)\bar{\mathcal{O}}_{[p,q,p]}(y)\rangle$. The only surviving contribution to $\langle \mathcal{K}_{[p,q,p]}\bar{\mathcal{K}}_{[p,q,p]}\rangle|_{g^2}$ is due to diagrams of type (c) again, and equals

$$\begin{aligned}
&\sum \mathrm{tr}\Big[\Big(t^{a_{\sigma(k+1)}}\dots t^{a_{\sigma(p+q)}}t^{b_1}\dots t^{b_{j-1}}[t^{a_i},t^c]t^{b_{j+1}}\dots t^{b_k}\Big)\Big(t^{a_{\sigma(1)}}\dots t^{a_{\sigma(k)}}t^{b_{k+1}}\dots t^{b_p}\Big)\Big]\times\\
&\times\mathrm{tr}\Big[\Big(t^{a_{l+1}}\dots t^{a_{p+q}}t^{b_{\rho(1)}}\dots t^{b_{\rho(l)}}\Big)\Big(t^{a_1}\dots t^{a_{i-1}}[t^{b_j},t^c]t^{a_{i+1}}\dots t^{a_l}t^{b_{\rho(l+1)}}\dots t^{b_{\rho(p)}}\Big)\Big]+\\
&+\sum \mathrm{tr}\Big[\Big(t^{a_{\sigma(k+1)}}\dots t^{a_{\sigma(p+q)}}t^{b_1}\dots t^{b_{j-1}}[t^{a_i},t^c]t^{b_{j+1}}\dots t^{b_k}\Big)\Big(t^{a_{\sigma(1)}}\dots t^{a_{\sigma(k)}}t^{b_{k+1}}\dots t^{b_p}\Big)\Big]\times\\
&\times\mathrm{tr}\Big[\Big(t^{a_{l+1}}\dots t^{a_{i-1}}[t^{b_j},t^c]t^{a_{i+1}}\dots t^{a_{p+q}}t^{b_{\rho(1)}}\dots t^{b_{\rho(l)}}\Big)\Big(t^{a_1}\dots t^{a_l}t^{b_{\rho(l+1)}}\dots t^{b_{\rho(p)}}\Big)\Big]+\\
&+\sum \mathrm{tr}\Big[\Big(t^{a_{\sigma(k+1)}}\dots t^{a_{\sigma(p+q)}}t^{b_1}\dots t^{b_k}\Big)\Big(t^{a_{\sigma(1)}}\dots t^{a_{\sigma(k)}}t^{b_{k+1}}\dots t^{b_{j-1}}[t^{a_i},t^c]t^{b_{j+1}}\dots t^{b_p}\Big)\Big]\times\\
&\times\mathrm{tr}\Big[\Big(t^{a_{l+1}}\dots t^{a_{p+q}}t^{b_{\rho(1)}}\dots t^{b_{\rho(l)}}\Big)\Big(t^{a_1}\dots t^{a_{i-1}}[t^{b_j},t^c]t^{a_{i+1}}\dots t^{a_l}t^{b_{\rho(l+1)}}\dots t^{b_{\rho(p)}}\Big)\Big]+\\
&+\sum \mathrm{tr}\Big[\Big(t^{a_{\sigma(k+1)}}\dots t^{a_{\sigma(p+q)}}t^{b_1}\dots t^{b_k}\Big)\Big(t^{a_{\sigma(1)}}\dots t^{a_{\sigma(k)}}t^{b_{k+1}}\dots t^{b_{j-1}}[t^{a_i},t^c]t^{b_{j+1}}\dots t^{b_p}\Big)\Big]\times\\
&\times\mathrm{tr}\Big[\Big(t^{a_{l+1}}\dots t^{a_{i-1}}[t^{b_j},t^c]t^{a_{i+1}}\dots t^{a_{p+q}}t^{b_{\rho(1)}}\dots t^{b_{\rho(l)}}\Big)\Big(t^{a_1}\dots t^{a_l}t^{b_{\rho(l+1)}}\dots t^{b_{\rho(p)}}\Big)\Big]
\end{aligned}\tag{9.18}$$

up to a factor of $(-1)\tilde{B}(x,y)[G(x,y)]^{(2p+q)}$, and proper symmetrizations. The sums on $\{i,j\}$ are as before: in the first line, from $\{1,1\}$ to $\{l,k\}$, in the second line, from $\{l+1,1\}$ to $\{p+q,k\}$, in the third line, from $\{1,k+1\}$ to $\{l,p\}$, and in the last line, from $\{l+1,k+1\}$ to $\{p+q,p\}$).

At Born level, $\langle \mathcal{K}_{[p,q,p]}\bar{\mathcal{K}}_{[p,q,p]}\rangle|_{\text{free}} \sim (\frac{1}{2}N)^{(2p+q)}$ at large N: we have to merge traces once, and there are $2p+q$ generators involved.

9.5 Quarter BPS operators

Given the leading large N dependence of a $\langle \mathcal{O}\bar{\mathcal{K}}\rangle|_{\text{free}}$ correlator, it's easy to determine the leading large N dependence of the corresponding $\langle \mathcal{K}\bar{\mathcal{K}}\rangle|_{\text{free}}$. Indeed, suppose that a particular term e.g.

$$\begin{aligned}
&(\mathrm{tr}\, t^a\dots t^a t^b\dots t^b)(\mathrm{tr}\, t^a\dots t^a t^b\dots t^b)\mathrm{tr}\,(t^a\dots t^a t^b\dots t^b)(t^a\dots t^a t^b\dots t^b)\sim\\
&\sim(2/N)(\mathrm{tr}\, t^a\dots t^a t^b\dots t^b)(\mathrm{tr}\, t^c t^c t^a\dots t^a t^b\dots t^b)\mathrm{tr}\,(t^a\dots t^a t^b\dots t^b)(t^a\dots t^a t^b\dots t^b)
\end{aligned}\tag{9.19}$$

contributes to $\langle\mathcal{O}\bar{\mathcal{K}}\rangle$. We can insert contents of the first and second traces into the third trace (using $2(\mathrm{tr}\,At^r)(\mathrm{tr}\,Bt^r)\sim\mathrm{tr}\,AB$) to reduce this expression to a single trace. On the other hand, the term in the $\langle\mathcal{K}\bar{\mathcal{K}}\rangle$ with the same order of generators can be written as

$$\begin{aligned}&\mathrm{tr}\,(t^a\dots t^a t^b\dots t^b)(t^a\dots t^a t^b\dots t^b)\mathrm{tr}\,(t^a\dots t^a t^b\dots t^b)(t^a\dots t^a t^b\dots t^b)\sim \\ &\sim 2(\mathrm{tr}\,t^a\dots t^a t^b\dots t^b t^c)(\mathrm{tr}\,t^c t^a\dots t^a t^b\dots t^b)\mathrm{tr}\,(t^a\dots t^a t^b\dots t^b)(t^a\dots t^a t^b\dots t^b)\end{aligned}\tag{9.20}$$

and we can insert the first and second traces into the third trace again in the same locations. This term gives a leading contribution provided all generators collapse after consecutively applying $2t^r t^r\sim N\mathbf{1}$ without having to commute generators past one another. In this case, the term in (9.19) also gives a leading contribution. The only other terms which have the same large N behavior are the ones that differ from it by cyclic permutations within the first and second traces. This gives an overall factor of $p(p+q)$. Comparing (9.19) and (9.20) then shows that to leading order in N, the difference is a factor of $\beta\equiv p(p+q)/N$.

For large N, the analysis of order g^2 corrections is analogous to the case of free field contributions. The structure of terms in (9.17) and (9.18) is similar, and leading contributions come from terms with the same order of generators (modulo cyclic permutations for $\langle\mathcal{O}\bar{\mathcal{K}}\rangle$ corrections); the difference is the multiplicative factor β, the same for all such terms.

Bringing this together with the results of sections 9.2–9.4, we can write down the large N leading order two point functions as

$$\begin{pmatrix}\langle\mathcal{K}\bar{\mathcal{K}}\rangle & \langle\mathcal{K}\bar{\mathcal{O}}\rangle\\ \langle\mathcal{O}\bar{\mathcal{K}}\rangle & \langle\mathcal{O}\bar{\mathcal{O}}\rangle\end{pmatrix}=\alpha(GN)^{2p+q}\left\{\begin{pmatrix}1&\beta\\ \beta&*\end{pmatrix}+\tilde{\alpha}(\tilde{B}N)\begin{pmatrix}1&\beta\\ \beta&\mathcal{O}(N^{-2})\end{pmatrix}\right\}\tag{9.21}$$

where α, $*$, and $\tilde{\alpha}$ are some constants of order $\mathcal{O}(N^0)$, and $\beta=p(p+q)/N$. As before, $G\equiv G(x,y)$; $\tilde{B}\equiv\tilde{B}(x,y)$; and $\langle\mathcal{O}\bar{\mathcal{O}}\rangle\equiv\langle\mathcal{O}_{[p,q,p]}(x)\bar{\mathcal{O}}_{[p,q,p]}(y)\rangle$, etc. Each (order one) coefficient above is valid to $\mathcal{O}(N^{-2})$, and of course (9.21) is perturbative in the coupling constant — we have neglected $\mathcal{O}(g^4)$ terms.

Diagonalizing the matrix of corrections with respect to the matrix of free correlators as in section 7, we find that

$$\tilde{\mathcal{Y}}_{[p,q,p]}=\mathcal{K}_{[p,q,p]}+\mathcal{O}(N^{-2})\tag{9.22}$$

$$\mathcal{Y}_{[p,q,p]}=\mathcal{O}_{[p,q,p]}-\frac{p(p+q)}{N}\mathcal{K}_{[p,q,p]}+\mathcal{O}(N^{-2})\tag{9.23}$$

are pure operators, and as such have well defined scaling dimension. At this order, the scaling dimension of $\tilde{\mathcal{Y}}_{[p,q,p]}$ receives an $\mathcal{O}(g^2N)$ correction proportional to the coefficient $\tilde{\alpha}$ in (9.21), while the scaling dimension $\Delta_{\mathcal{Y}}=2p+q$ of $\mathcal{Y}_{[p,q,p]}$ is protected. Finally, the normalization of $\mathcal{Y}_{[p,q,p]}$ does not get any g^2 corrections, and is given by the Born level expression

$$\langle \mathcal{Y}_{[p,q,p]}(x)\bar{\mathcal{Y}}_{[p,q,p]}(y)\rangle = [1+\delta_{q,0}(-)^p]\,\frac{p(p+q)(p+q+1)}{(q+1)}\left[\frac{N}{8\pi(x-y)^2}\right]^{2p+q}\times$$
$$\times\left(1+\mathcal{O}(N^{-2};g^4)\right)\ . \tag{9.24}$$

This is found from formulae (9.8), (9.21), and (9.23). The exact expressions of sections 5 and 7 agree with (9.23) and (9.24) in the large N limit. We conclude that to working precision, $\mathcal{Y}_{[p,q,p]}$ is a $\frac{1}{4}$-BPS chiral primary operator.

10. Conclusion

In this paper we studied local, polynomial, gauge invariant scalar composite operators in $[p,q,p]$ representations of SU(4) in $\mathcal{N}$=4 SYM in four dimensions. We found that certain such operators have protected two-point functions at order g^2, with each other as well as with other operators. We presented ample evidence that these $\mathcal{O}(g^2)$ protected operators are $\frac{1}{4}$-BPS chiral primaries in the fully interacting theory.

These operators are not just the double trace operators from the classification of [2], but mixtures of all gauge invariant local composite operators made of the same scalars: single trace operators, other double trace operators, triple trace operators, etc. As our exact in N, explicit construction of $\frac{1}{4}$-BPS primaries of low scaling dimension ($\Delta = 2p+q < 8$) shows, the N dependence of the coefficients in these linear combinations is quite complicated.

Apart from operators of low dimensions for arbitrary N, we considered the large N behavior of two point functions for all $[p,q,p]$. A leading and subleading analysis reveals that for every $[p,q,p]$, there is a $\frac{1}{4}$-BPS operator which is a certain linear combinations of double and single trace operators. We give closed form expressions for the operators involved in this linear combination and the coefficients with which they enter, all valid to next-to-leading order in N.

Acknowledgments

I would like to thank Sergio Ferrara for useful discussions, as well as pointing out references [8]. I am especially grateful to Eric D'Hoker, who suggested the problem and without whose involvement this paper would not be finished.

A. $\mathcal{N}$= 4 SUSY in various forms

In the $\mathcal{N}$=1 component notation, the classical lagrangian (see [21, p. 158]) takes the form (in geometric notation, i.e. with $\frac{1}{g^2}$ multiplying the whole action)

$$\begin{aligned}\mathcal{L} = \frac{1}{g^2}\mathrm{tr}\Big\{ & -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{i}{2}\bar{\lambda}\gamma^\mu D_\mu\lambda + \frac{1}{2}D^2 + \frac{1}{2}D_\mu A_j D^\mu A_j + \frac{1}{2}D_\mu B_j D^\mu B_j + \\ & + \frac{i}{2}\bar{\psi}_j\gamma^\mu D_\mu\psi_j + \frac{1}{2}F_jF_j + \frac{1}{2}G_jG_j - i[A_j, B_j]Di\bar{\psi}_j[\lambda, A_j]i\bar{\psi}_j - \\ & - \gamma_5[\lambda, B_j] - \frac{i}{2}\epsilon_{jkl}\Big(\bar{\psi}_j[\psi_k, A_l] - \bar{\psi}_j\gamma_5[\psi_k, B_l] + [A_j, A_k]F_l - \\ & - [B_j, B_k]F_l + 2[A_j, B_k]G_l\Big)\Big\}\end{aligned} \tag{A.1}$$

(with Lorentz signature); there should be no confusion between the auxiliary field D of the vector multiplet and the covariant derivative $D_\mu = \partial_\mu + iA_\mu$.

This lagrangian can also be rewritten in a manifestly SU(4)-invariant form. Combining the three chiral spinors and the gaugino into

$$\lambda_4 = \lambda\,; \qquad \lambda_j = \psi_j\,, \quad j = 1, 2, 3 \tag{A.2}$$

and making 4×4 antisymmetric matrices of scalars and pseudoscalars by

$$A_{jk} = -\epsilon_{jkl}A_l; \qquad A_{4j} = -A_{j4} = A_j; \tag{A.3}$$

$$B_{jk} = \quad \epsilon_{jkl}B_l; \qquad B_{4j} = -B_{j4} = B_j \tag{A.4}$$

the lagrangian becomes (sums on indices j, k, l now run from 1 to 4)

$$\begin{aligned}\mathcal{L} = \frac{1}{g^2}\mathrm{tr}\Big\{ & -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{i}{2}\bar{\lambda}_j\gamma^\mu D_\mu\lambda_j + \frac{1}{2}D_\mu A_{jk}D^\mu A_{jk} + \frac{1}{2}D_\mu B_{jk}D^\mu B_{jk} + \\ & + \frac{i}{2}\bar{\lambda}_j[\lambda_k, A_{jk}] + \frac{i}{2}\bar{\lambda}_j\gamma_5[\lambda_k, B_{jk}] + \frac{1}{32}[A_{jk}, B_{lm}][A_{jk}, B_{lm}] + \\ & + \frac{1}{64}[A_{jk}, A_{lm}][A_{jk}, A_{lm}] + \frac{1}{64}[B_{jk}, B_{lm}][B_{jk}, B_{lm}]\Big\}\end{aligned} \tag{A.5}$$

after integrating out the auxiliary fields D, F_j, and G_j. The A_{jk} and B_{jk} are self-dual and antiself-dual tensors of $O(4)$:

$$A_{jk} = \frac{1}{2}\epsilon_{jklm}A_{lm}\,; \quad B_{jk} = -\frac{1}{2}\epsilon_{jklm}B_{lm}\,. \tag{A.6}$$

Alternatively, the fields A_i and B_i form a **6** of the R-symmetry group SU(4) $\sim$ SO(6): we can group them as $\phi^i = A_i$, $\phi^{i+3} = B_i$, $i = 1, 2, 3$.

This form of the lagrangian is only manifestly $O(4)$ symmetric, however. If we define a complex matrix of scalars

$$M_{jk} \equiv \frac{1}{2}\left(A_{lm} + iB_{jk}\right) \tag{A.7}$$

subject to a reality condition

$$\bar{M}^{jk} \equiv (M_{jk})^\dagger = \frac{1}{2}\epsilon^{jklm}M_{lm} \tag{A.8}$$

the $\mathcal{N}$=4 lagrangian

$$\begin{aligned}\mathcal{L} = \frac{1}{g^2}\mathrm{tr}\Big\{ & -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + i\lambda_j\sigma^\mu D_\mu\bar{\lambda}^j + \frac{1}{2}D_\mu M_{jk}\bar{D}^\mu\bar{M}^{jk} + \\ & + i\lambda_j[\lambda_k,\bar{M}^{jk}] + i\bar{\lambda}^j[\bar{\lambda}^k,M_{jk}] + \frac{1}{4}[M_{jk},M_{lm}][\bar{M}^{jk},\bar{M}^{lm}]\Big\}\end{aligned} \tag{A.9}$$

is then manifestly SU(4) covariant, as are the SUSY transformation laws

$$\begin{aligned}\delta A_\mu &= i\zeta_j\sigma_\mu\bar{\lambda}^j - i\lambda_j\sigma_\mu\bar{\zeta}^j \\ \delta M_{jk} &= \zeta_j\lambda_k - \zeta_k\lambda_j + \epsilon_{jklm}\bar{\zeta}^l\bar{\lambda}^m \\ \delta\lambda_j &= -\frac{i}{2}\sigma^{\mu\nu}F_{\mu\nu}\zeta_j + 2i\sigma^\mu D_\mu M_{jk}\bar{\zeta}^k + 2i[M_{jk},\bar{M}^{kl}]\zeta_l\end{aligned} \tag{A.10}$$

(notice that now λ_j and $\bar{\lambda}^j$ are Weyl spinors; there should be no confusion: when spinors are multiplied by 2×2 σ matrices they are Weyl, and when the 4×4 γ matrices are used, they are Dirac).

We can also rewrite this lagrangian in terms of three unconstrained (unlike the M_{jk}) complex scalar fields

$$z_j = \frac{1}{\sqrt{2}}\left(A_j + iB_j\right), \quad \bar{z}^j = \frac{1}{\sqrt{2}}\left(A_j - iB_j\right) \tag{A.11}$$

and the original fermions ψ_j and λ:

$$\begin{aligned}\mathcal{L} = \frac{1}{g^2}\mathrm{tr}\Big\{ & -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + i\lambda\sigma^\mu D_\mu\bar{\lambda} + i\psi_j\sigma^\mu D_\mu\bar{\psi}^j + D_\mu z_j\bar{D}^\mu\bar{z}^j + \\ & + \frac{i}{\sqrt{2}}\lambda[\psi_j,\bar{z}^j] - \frac{i}{\sqrt{2}}\psi_j[\lambda,\bar{z}^j] - \frac{i}{\sqrt{2}}\epsilon^{jkl}\psi_j[\psi_k,z_l] + \\ & + \frac{i}{\sqrt{2}}\bar{\lambda}[\bar{\psi}^j,z_j] - \frac{i}{\sqrt{2}}\bar{\psi}^j[\bar{\lambda},z_j] - \frac{i}{\sqrt{2}}\epsilon_{jkl}\bar{\psi}^j[\bar{\psi}^k,\bar{z}^l] + \\ & + [z_j,z_k][\bar{z}^j,\bar{z}^k] - \frac{1}{2}[z_j,\bar{z}^j][z_k,\bar{z}^k]\Big\}\end{aligned} \tag{A.12}$$

and the SUSY transformations now are

$$\begin{aligned}\delta A_\mu &= i\zeta_j\sigma_\mu\bar{\psi}^j - i\psi_j\sigma_\mu\bar{\zeta}^j + i\zeta\sigma_\mu\bar{\lambda} - i\lambda\sigma_\mu\bar{\zeta} \\ \delta z_j &= \sqrt{2}\left(\zeta\psi_j - \zeta_j\lambda - \epsilon_{jkl}\bar{\zeta}^k\bar{\psi}^l\right)\end{aligned} \tag{A.13}$$

$$\delta\lambda = -\frac{i}{2}\sigma^{\mu\nu}F_{\mu\nu}\zeta + i\sqrt{2}\sigma^\mu D_\mu z_j\bar{\zeta}^j + i\epsilon^{jkl}[z_j,z_k]\zeta_l - i[z_j,\bar{z}^j]\zeta \tag{A.14}$$

$$\begin{aligned}\delta\psi_j = & -\frac{i}{2}\sigma^{\mu\nu}F_{\mu\nu}\zeta_j + i\sqrt{2}\epsilon_{jkl}\sigma^\mu\bar{D}_\mu\bar{z}^k\bar{\zeta}^l - i\sqrt{2}\sigma^\mu D_\mu z_j\bar{\zeta} + \\ & + i\left([z_k,\bar{z}^k]\zeta_j - 2[z_j,\bar{z}^k]\zeta_k - \epsilon_{jkl}[\bar{z}^k,\bar{z}^l]\zeta\right)\end{aligned} \tag{A.15}$$

and their conjugates

$$\delta\bar{z}^j = \sqrt{2}\left(\bar{\zeta}\bar{\psi}^j - \bar{\zeta}^j\bar{\lambda} - \epsilon^{jkl}\zeta_k\psi_l\right) \tag{A.16}$$

$$\delta\bar{\lambda} = +\frac{i}{2}\bar{\sigma}^{\mu\nu}F_{\mu\nu}\bar{\zeta} - i\sqrt{2}\bar{\sigma}^{\mu}\bar{D}_{\mu}\bar{z}^j\zeta_j - i\epsilon_{jkl}[\bar{z}^j,\bar{z}^k]\bar{\zeta}^l + i[\bar{z}^j, z_j]\bar{\zeta} \tag{A.17}$$

$$\begin{aligned}\delta\bar{\psi}^j &= +\frac{i}{2}\bar{\sigma}^{\mu\nu}F_{\mu\nu}\bar{\zeta}^j - i\sqrt{2}\epsilon^{jkl}\bar{\sigma}^{\mu}D_{\mu}z_k\zeta_l + i\sqrt{2}\bar{\sigma}^{\mu}\bar{D}_{\mu}\bar{z}^j\zeta \\ &\quad -i\left([\bar{z}^k, z_k]\bar{\zeta}^j - 2[\bar{z}^j, z_k]\bar{\zeta}^k - \epsilon^{jkl}[z_k, z_l]\bar{\zeta}\right).\end{aligned} \tag{A.18}$$

This way of writing the lagrangian and SUSY transformations hides the full SU(4) R-symmetry of the theory; now, only the $\mathrm{SU}(3)\times\mathrm{U}(1)$ subgroup of it is manifest.

B. Miscellaneous identities for $\mathrm{SU}(N)$

We can use the following property of generators of $\mathrm{SU}(N)$ (for $N \geq 3$) in the fundamental representation:

$$\{t^a, t^b\} = \frac{1}{N}\delta^{ab} + d^{abc}t^c\,. \tag{B.1}$$

Together with $[t^a, t^b] = if^{abc}t^c$ (valid in any representation), we find

$$t^a t^b = \frac{1}{2N}\delta^{ab}\mathbf{1} + \frac{1}{2}\left(d^{abc} + if^{abc}\right)t^c\,. \tag{B.2}$$

Let

$$g^{a_1\ldots a_k} \equiv \mathrm{tr}\, t^{a_1}\ldots t^{a_k}\,. \tag{B.3}$$

Then with the standard normalization $\mathrm{tr}\, t^a t^b = 1/2\delta^{ab}$ for $\mathrm{SU}(N)$ generators in the fundamental, we can in principle recursively determine the trace of any string of generators in terms of δ^{ab}, d^{abc}, and f^{abc}:

$$\begin{aligned} g^a &= \mathrm{tr}\, t^a = 0\,,\quad g^{ab} = \frac{1}{2}\delta^{ab}\,,\quad g^{abc} = \frac{1}{4}\left(d^{abc} + if^{abc}\right),\quad \text{and} \\ g^{a_1\ldots a_k} &= \frac{1}{2N}\,\delta^{a_1a_2}g^{a_3\ldots a_k} + 2\,g^{a_1a_2c}g^{ca_3\ldots a_k}\end{aligned} \tag{B.4}$$

(for completeness we can define $g^{(0)} = \mathrm{tr}\,\mathbf{1} = N$).

Now we can set up a recursion relation for

$$P_k \equiv g^{a_1\ldots a_k}g^{a_1\ldots a_k} \tag{B.5}$$

and

$$\tilde{P}_k \equiv g^{a_1\ldots a_k}g^{a_k\ldots a_1} \tag{B.6}$$

(with sums on repeated $a_1,\ldots,a_k$ implied). Using $t^a t^a = (N^2-1)/2N\mathbf{1}$, we find

$$P_k = \frac{N^2-1}{4N^2}P_{k-2} + \frac{4}{N^2-1}P_3P_{k-1} \tag{B.7}$$

and similarly

$$\tilde{P}_k = \frac{N^2-1}{4N^2}\tilde{P}_{k-2} + \frac{4}{N^2-1}\tilde{P}_3\tilde{P}_{k-1}\,. \tag{B.8}$$

The values of P_2, $\tilde{P}_2$, P_3 and $\tilde{P}_3$ have to be computed explicitly; they are

$$P_2 = \tilde{P}_2 = \frac{N^2-1}{4}\,, \quad P_3 = -\frac{N^2-1}{4N}\,, \quad \text{and} \quad \tilde{P}_3 = \frac{(N^2-1)(N^2-2)}{8N}\,. \tag{B.9}$$

For large N, the leading behavior is given by

$$g^{a_1\ldots a_k} \sim \left(2^{k-3}\right) g^{a_1a_2c_3}g^{c_3a_3c_4}\ldots g^{c_{k-2}a_{k-2}c_{k-1}}g^{c_{k-1}a_{k-1}a_k} \tag{B.10}$$

and

$$P_{2k+1} \sim -\frac{Nk}{4^k}, \quad P_{2k} \sim \frac{N^2}{4^k}; \quad \tilde{P}_k \sim \frac{N^k}{2^k}\,. \tag{B.11}$$

Dependence on N is[28] very different for P_k and $\tilde{P}_k$; in fact, taking generators in reverse order in the second trace (such as in $\tilde{P}_k$) grows the fastest with k, and taking them in the same order (as in P_k), the slowest.

Here are a few more identities we may have a need for in calculating two-point functions. First, the normalizations of $\mathrm{SU}(N)$ generators in an arbitrary representation is defined in terms of a constant $C(r)$ as

$$\mathrm{tr}_r\, T_r^a T_r^b = C(r)\delta^{ab} \tag{B.12}$$

and there is a quadratic Casimir,

$$T_r^c T_r^c = C_2(r)\,\mathbf{1}\,. \tag{B.13}$$

In particular, the adjoint and fundamental representations will be of interested, and for these $C_2(\text{adj}) = N$, $C_2(\text{fund}) = (\frac{N^2-1}{2N})$, $C(\text{adj}) = N$, $C(\text{fund}) = \frac{1}{2}$. Then, for example,

$$T^aT^aT^bT^b = \left[C_2(r)\right]^2\ \mathbf{1} \tag{B.14}$$

$$T^aT^bT^bT^a = \left[C_2(r)\right]^2\ \mathbf{1} \tag{B.15}$$

$$\left[T^a, T^b\right]\left[T^a, T^b\right] = -NC_2(r)\ \mathbf{1} \tag{B.16}$$

$$T^aT^bT^aT^b = C_2(r)\left(C_2(r) - \frac{N}{2}\right)\ \mathbf{1} \tag{B.17}$$

(we have omitted the label "r" on the generators, e.g. $T^a = T_r^a$). Longer expressions are just a little more complicated but not by much:

$$\mathrm{tr}\, T^b\left[T^a, T^b\right]T^c\left[T^a, T^c\right] = \frac{1}{4}N^2(N^2-1)C(r) \tag{B.18}$$

$$\mathrm{tr}\, T^aT^bT^cT^aT^cT^b = (N^2-1)C(r)\left(C_2(r) - \frac{N}{2}\right)^2 \tag{B.19}$$

$$\mathrm{tr}\, T^aT^bT^cT^aT^bT^c = (N^2-1)C(r)\left(C_2(r) - \frac{N}{2}\right)\left(C_2(r) - N\right). \tag{B.20}$$

[28]Note that the way the recursion formulae (B.6) and (B.6) work out together with the initial values (B.9), leading order large N results are accurate to order $\mathcal{O}(N^{-2})$ and not to $\mathcal{O}(N^{-1})$, as one could have thought naively.

In particular, the last expression vanishes in the adjoint representation.

Using the fact that $\mathrm{U}(N)_C = \mathrm{Gl}(N,C)$, any $N \times N$ matrix A can be decomposed into generators (in the fundamental) of $\mathrm{SU}(N)$ plus the unit matrix:

$$A = (2\mathrm{tr}\, A t^c)\, t^c + \left(\frac{1}{N}\mathrm{tr}\, A\right) \mathbf{1}\,. \tag{B.21}$$

Then, for example, we can write down the "trace merging formula"

$$2\,(\mathrm{tr}\, A t^c)\,(\mathrm{tr}\, B t^c) = \mathrm{tr}\, AB - \frac{1}{N}\,(\mathrm{tr}\, A)\,(\mathrm{tr}\, B) \tag{B.22}$$

and we can arrive at an even simpler recursion relations for $\tilde{P}_k$:

$$\begin{aligned} \tilde{P}_{k+1} &= (\mathrm{tr}\, t^{a_1} \dots t^{a_k} t^c)\,(\mathrm{tr}\, t^{a_k} \dots t^{a_1} t^c) \\ &= \frac{1}{2}\,(\mathrm{tr}\, t^{a_1} \dots t^{a_k} t^{a_k} \dots t^{a_1}) - \frac{1}{2N}\,(\mathrm{tr}\, t^{a_1} \dots t^{a_k})\,(\mathrm{tr}\, t^{a_k} \dots t^{a_1}) \\ &= \frac{N}{2}\left(\frac{N^2-1}{2N}\right)^k - \frac{1}{2N}\tilde{P}_k \end{aligned} \tag{B.23}$$

with $\tilde{P}_1 = 0$. (Naturally, this gives the same values for $\tilde{P}_k$ as before.)

Another useful relation satisfied by the generators of $\mathrm{SU}(N)$ in the fundamental, is

$$(t^a)_{ij}(t^a)_{kl} = \frac{1}{2}\left(\delta_{il}\delta_{jk} - \frac{1}{N}\delta_{ij}\delta_{kl}\right). \tag{B.24}$$

Using this identity, one can easily reproduce the trace merging formula (B.22), as well as the expressions (B.9) for P_2, $\tilde{P}_2$, P_3 and $\tilde{P}_3$.

C. Details for operators of dimension $\Delta \geq 6$

C.1 States with weight [2,2,2]

Instead of computing the symmetry factors by hand, we fed *Mathematica* an algorithm for calculating the contractions for the free correlators as well as the $\mathcal{O}(g^2)$ corrections; the results are

$$\begin{aligned} &\frac{192}{5N^4(N^2-1)}\,\mathbf{F} = \\ &= \begin{pmatrix} 1-\frac{4}{N^2}+\frac{40}{3N^4} & \frac{8(3N^4-22N^2+60)}{3N^5} & \frac{4(N^2-4)(N^2-5)}{N^5} & \frac{2(7N^4-18N^2+40)}{N^5} & \frac{20(N^2-2)}{N^4} \\ \frac{8(3N^4-22N^2+60)}{3N^5} & 8-\frac{152}{3N^2}+\frac{224}{N^4}-\frac{384}{N^6} & \frac{-8(N^2-4)(N^2-6)}{N^6} & \frac{4(N^6+7N^4+8N^2-48)}{N^6} & \frac{16(2N^4-7N^2+6)}{N^5} \\ \frac{4(N^2-4)(N^2-5)}{N^5} & \frac{-8(N^2-4)(N^2-6)}{N^6} & \frac{3(N^2-4)^2(N^2-2)}{N^6} & \frac{6(N^4-16)}{N^6} & \frac{-12(N^2-4)}{N^5} \\ \frac{2(7N^4-18N^2+40)}{N^5} & \frac{4(N^6+7N^4+8N^2-48)}{N^6} & \frac{6(N^4-16)}{N^6} & \frac{12(1+N^2)(N^4+6N^2-8)}{N^6} & \frac{12(3N^4-3N^2+4)}{N^5} \\ \frac{20(N^2-2)}{N^4} & \frac{16(2N^4-7N^2+6)}{N^5} & \frac{-12(N^2-4)}{N^5} & \frac{12(3N^4-3N^2+4)}{N^5} & \frac{12(N^2-2)(N^2+1)}{N^4} \end{pmatrix} \end{aligned} \tag{C.1}$$

for the matrix of free combinatorial factors, and

$$\frac{96}{25N^4(N^2-1)}\,\mathbf{G} = \tag{C.2}$$

$$= \begin{pmatrix} 1+\frac{4}{N^2}+\frac{8}{N^4} & \frac{8(N^4-2N^2+12)}{N^5} & \frac{3(N^2-4)^2}{N^5} & \frac{12(2N^4+N^2+4)}{N^5} & \frac{12(3N^2-2)}{N^4} \\ \frac{8(N^4-2N^2+12)}{N^5} & \frac{64(N^4-6N^2+18)}{N^6} & \frac{24(N^2-4)(N^2-6)}{N^6} & \frac{48(3N^4-5N^2+12)}{N^6} & \frac{96(2N^2-3)}{N^5} \\ \frac{3(N^2-4)^2}{N^5} & \frac{24(N^2-4)(N^2-6)}{N^6} & \frac{18(N^2-4)^2}{N^6} & \frac{18(N^2-4)^2}{N^6} & \frac{36(N^2-4)}{N^5} \\ \frac{12(2N^4+N^2+4)}{N^5} & \frac{48(3N^4-5N^2+12)}{N^6} & \frac{18(N^2-4)^2}{N^6} & \frac{18(N^6+17N^4-8N^2+16)}{N^6} & \frac{36(N^4+7N^2-4)}{N^5} \\ \frac{12(3N^2-2)}{N^4} & \frac{96(2N^2-3)}{N^5} & \frac{36(N^2-4)}{N^5} & \frac{36(N^4+7N^2-4)}{N^5} & \frac{72(N^2+1)}{N^4} \end{pmatrix}$$

for the matrix of corrections proportional to $\tilde{B}(x,y)N$; for the notation see section 7.

C.2 States with weight [2,3,2]

In the same fashion, for the operators defined in section 7.3 (also of scaling dimension $\Delta = 7 + \mathcal{O}(g^2)$), we find

$$\frac{64}{3\,(N^2-1)\,(N^2-4)\,N^3}\,\mathbf{F} = \tag{C.3}$$

$$= \left(\begin{array}{cccc} \frac{(2N^2+15)}{2N^2} & \frac{10(N^4-3N^2+30)}{N^5} & \frac{5(7N^4-3N^2+30)}{N^5} & \frac{15(N^4-8N^2+30)}{N^5} \\ \frac{10(N^4-3N^2+30)}{N^5} & \frac{10(N^6+12N^2-60)}{N^6} & \frac{5(N^6+45N^4-78N^2-60)}{N^6} & \frac{30(N^2-6)(N^2+5)}{N^6} \\ \frac{5(7N^4-3N^2+30)}{N^5} & \frac{5(N^6+45N^4-78N^2-60)}{N^6} & \frac{10(4N^6+45N^4-42N^2-15)}{N^6} & \frac{30(8N^4-23N^2-15)}{N^6} \\ \frac{15(N^4-8N^2+30)}{N^5} & \frac{30(N^2-6)(N^2+5)}{N^6} & \frac{30(8N^4-23N^2-15)}{N^6} & \frac{15(N^2-3)(N^4-6N^2+30)}{N^6} \\ \frac{15(5N^4-16N^2+60)}{2N^5} & \frac{30(2N^2+3)(3N^2-10)}{N^6} & \frac{30(18N^4-28N^2-15)}{N^6} & \frac{15(N^6-4N^4+18N^2-90)}{N^6} \\ \frac{30(9N^2-25)}{N^4} & \frac{30(N^2-2)(11N^2-25)}{N^5} & \frac{30(13N^4+14N^2+25)}{N^5} & \frac{30(2N^4-17N^2+75)}{N^5} \\ -\frac{15(3N^2-10)}{N^4} & -\frac{30(N^2-2)(4N^2-5)}{N^5} & \frac{-30(2N^4+N^2+5)}{N^5} & \frac{30(2N^4-2N^2-15)}{N^5} \end{array} \right.$$

$$\left. \begin{array}{ccc} \frac{15(5N^4-16N^2+60)}{2N^5} & \frac{30(9N^2-25)}{N^4} & -\frac{15(3N^2-10)}{N^4} \\ \frac{30(2N^2+3)(3N^2-10)}{N^6} & \frac{30(N^2-2)(11N^2-25)}{N^5} & -\frac{30(N^2-2)(4N^2-5)}{N^5} \\ \frac{30(18N^4-28N^2-15)}{N^6} & \frac{30(13N^4+14N^2+25)}{N^5} & -\frac{30(2N^4+N^2+5)}{N^5} \\ \frac{15(N^6-4N^4+18N^2-90)}{N^6} & \frac{30(2N^4-17N^2+75)}{N^5} & \frac{30(2N^4-2N^2-15)}{N^5} \\ \frac{15(7N^6+32N^4-24N^2-180)}{2N^6} & \frac{30(7N^4+8N^2+75)}{N^5} & -\frac{15(N^4+14N^2+30)}{N^5} \\ \frac{30(7N^4+8N^2+75)}{N^5} & \frac{30(N^2+5)(11N^2-25)}{N^4} & -\frac{30(N^2+5)(4N^2-5)}{N^4} \\ -\frac{15(N^4+14N^2+30)}{N^5} & -\frac{30(N^2+5)(4N^2-5)}{N^4} & \frac{15(7N^4+15N^2-10)}{N^4} \end{array} \right)$$

for the matrix of free combinatorial factors, and

$$\frac{16}{27\,(N^2-1)\,(N^2-4)\,N^3}\mathbf{G} =$$

$$
= \begin{pmatrix}
\frac{N^2+29}{N^2} & \frac{10(N^2+12)}{N^3} & \frac{10(5N^2+21)}{N^3} & \frac{10(N^2+3)}{N^3} & \frac{60(N^2+3)}{N^3} & \frac{420}{N^2} & -\frac{90}{N^2} \\
\frac{10(N^2+12)}{N^3} & \frac{100(N^4+24)}{N^6} & \frac{400(N^4+3)}{N^6} & \frac{100(N^4-6N^2+36)}{N^6} & \frac{150(3N^4-4N^2+24)}{N^6} & \frac{3000(N^2-2)}{N^5} & \frac{-600(N^2-2)}{N^5} \\
\frac{10(5N^2+21)}{N^3} & \frac{400(N^4+3)}{N^6} & \frac{50(N^6+19N^4+12)}{N^6} & \frac{300(N^4-N^2+6)}{N^6} & \frac{300(4N^4-N^2+6)}{N^6} & \frac{300(N^4+17N^2-10)}{N^5} & -\frac{600(2N^2-1)}{N^5} \\
\frac{10(N^2+3)}{N^3} & \frac{100(N^4-6N^2+36)}{N^6} & \frac{300(N^4-N^2+6)}{N^6} & \frac{200(N^4-9N^2+27)}{N^6} & \frac{300(N^4-6N^2+18)}{N^6} & \frac{600(4N^2-15)}{N^5} & -\frac{300(N^2-6)}{N^5} \\
\frac{60(N^2+3)}{N^3} & \frac{150(3N^4-4N^2+24)}{N^6} & \frac{300(4N^4-N^2+6)}{N^6} & \frac{300(N^4-6N^2+18)}{N^6} & \frac{75(N^6+17N^4-24N^2+72)}{N^6} & \frac{300(N^4+20N^2-30)}{N^5} & -\frac{150(N^4+8N^2-12)}{N^5} \\
\frac{420}{N^2} & \frac{3000(N^2-2)}{N^5} & \frac{300(N^4+17N^2-10)}{N^5} & \frac{600(4N^2-15)}{N^5} & \frac{300(N^4+20N^2-30)}{N^5} & \frac{3000(N^2+5)}{N^4} & -\frac{600(N^2+5)}{N^4} \\
-\frac{90}{N^2} & \frac{-600(N^2-2)}{N^5} & -\frac{600(2N^2-1)}{N^5} & -\frac{300(N^2-6)}{N^5} & -\frac{150(N^4+8N^2-12)}{N^5} & -\frac{600(N^2+5)}{N^4} & \frac{300(N^2+2)}{N^4}
\end{pmatrix} \tag{C.4}
$$

for the matrix of corrections proportional to $\tilde{B}(x,y)N$.

References

[1] V.K. Dobrev and V.B. Petkova, *All positive energy unitary irreducible representations of extended conformal supersymmetry*, *Phys. Lett.* **B 162** (1985) 127.

[2] L. Andrianopoli, S. Ferrara, E. Sokatchev and B. Zupnik, *Shortening of primary operators in n-extended sCFT$_4$ and harmonic-superspace analyticity*, *Adv. Theor. Math. Phys.* **3** (1999) 1149 [hep-th/9912007];
L. Andrianopoli and S. Ferrara, *Short and long* SU(2, 2/4) *multiplets in the AdS/CFT correspondence*, *Lett. Math. Phys.* **48** (1999) 145 [hep-th/9812067].

[3] J. Maldacena, *The large-N limit of superconformal field theories and supergravity*, *Adv. Theor. Math. Phys.* **2** (1998) 231 [hep-th/9711200];
S.S. Gubser, I.R. Klebanov and A.M. Polyakov, *Gauge theory correlators from non-critical string theory*, *Phys. Lett.* **B 428** (1998) 105 [hep-th/9802109];
E. Witten, *Anti-de Sitter space and holography*, *Adv. Theor. Math. Phys.* **2** (1998) 253 [hep-th/9802150].

[4] S.-M. Lee, S. Minwalla, M. Rangamani and N. Seiberg, *Three-point functions of chiral operators in $D = 4$, $N = 4$ SYM at large N*, *Adv. Theor. Math. Phys.* **2** (1998) 697 [hep-th/9806074].

[5] N. Dorey, T.J. Hollowood, V.V. Khoze, M.P. Mattis and S. Vandoren, *Multi-instantons and Maldacena's conjecture*, *J. High Energy Phys.* **06** (1999) 023 [hep-th/9810243]; *Multi-instanton calculus and the AdS/CFT correspondence in $N = 4$ superconformal field theory*, *Nucl. Phys.* **B 552** (1999) 88 [hep-th/9901128].

[6] E. D'Hoker, D.Z. Freedman and W. Skiba, *Field theory tests for correlators in the AdS/CFT correspondence*, *Phys. Rev.* **D 59** (1999) 045008 [hep-th/9807098].

[7] P.S. Howe, E. Sokatchev and P.C. West, *3-point functions in $N = 4$ Yang-Mills*, *Phys. Lett.* **B 444** (1998) 341 [hep-th/9808162];
B. Eden, P.S. Howe and P.C. West, *Nilpotent invariants in $N = 4$ SYM*, *Phys. Lett.* **B 463** (1999) 19 [hep-th/9905085].

[8] M. Bianchi, S. Kovacs, G. Rossi and Y.S. Stanev, *On the logarithmic behavior in $N = 4$ SYM theory*, *J. High Energy Phys.* **08** (1999) 020 [hep-th/9906188]; *Anomalous dimensions in $N = 4$ SYM theory at order g^4*, *Nucl. Phys.* **B 584** (2000) 216 [hep-th/0003203]; *Properties of the konishi multiplet in $N = 4$ sym theory*, *J. High Energy Phys.* **05** (2001) 042 [hep-th/0104016].

[9] G. Arutyunov and S. Frolov, *Some cubic couplings in type-IIB supergravity on $AdS_5 \times S^5$ and three-point functions in SYM$_4$) at large-N*, *Phys. Rev.* **D 61** (2000) 064009 [hep-th/9907085];
E. D'Hoker, S.D. Mathur, A. Matusis and L. Rastelli, *The operator product expansion of $N = 4$ SYM and the 4-point functions of supergravity*, *Nucl. Phys.* **B 589** (2000) 38 [hep-th/9911222];
G. Arutyunov and S. Frolov, *Scalar quartic couplings in type-IIB supergravity on $AdS_5 \times S^5$*, *Nucl. Phys.* **B 579** (2000) 117 [hep-th/9912210]; *Four-point functions of lowest weight cpos in $N = 4$ SYM$_4$ in supergravity approximation*, *Phys. Rev.* **D 62** (2000) 064016 [hep-th/0002170];
G. Arutyunov, S. Frolov and A.C. Petkou, *Operator product expansion of the lowest weight cpos in $N = 4$ SYM$_4$ at strong coupling*, *Nucl. Phys.* **B 586** (2000) 547 [hep-th/0005182]; *Perturbative and instanton corrections to the OPE of CPOS in $N = 4$ SYM$_4$*, *Nucl. Phys.* **B 602** (2001) 238 [hep-th/0010137].

[10] S. Penati, A. Santambrogio and D. Zanon, *More on correlators and contact terms in $N = 4$ SYM at order g^4*, *Nucl. Phys.* **B 593** (2001) 651 [hep-th/0005223].

[11] K.A. Intriligator, *Bonus symmetries of $N = 4$ super-Yang-Mills correlation functions via AdS duality*, *Nucl. Phys.* **B 551** (1999) 575 [hep-th/9811047];
K.A. Intriligator and W. Skiba, *Bonus symmetry and the operator product expansion of $N = 4$ super-Yang-Mills*, *Nucl. Phys.* **B 559** (1999) 165 [hep-th/9905020].

[12] B. Eden, P.S. Howe and P.C. West, *Nilpotent invariants in $N = 4$ SYM*, *Phys. Lett.* **B 463** (1999) 19 [hep-th/9905085].

[13] S.-M. Lee, *AdS_5/CFT_4 four-point functions of chiral primary operators: cubic vertices*, *Nucl. Phys.* **B 563** (1999) 349 [hep-th/9907108].

[14] G. Arutyunov, B. Eden and E. Sokatchev, *On non-renormalization and OPE in superconformal field theories*, hep-th/0105254;
B. Eden and E. Sokatchev, *On the OPE of 1/2 BPS short operators in $N = 4$ $sCFT_4$*, *Nucl. Phys.* **B 618** (2001) 259 [hep-th/0106249];
P.J. Heslop and P.S. Howe, *Opes and 3-point correlators of protected operators in $N = 4$ SYM*, hep-th/0107212.

[15] W. Skiba, *Correlators of short multi-trace operators in $N = 4$ supersymmetric Yang-Mills*, *Phys. Rev.* **D 60** (1999) 105038 [hep-th/9907088].

[16] J. Collins, *Renormalization*, chapter 4, Cambridge University Press, Cambridge 1984.

[17] H. Goldstein, *Classical mechanics*, chapter 6, Addison-Wesley 1980.

[18] M. Hamermesh, *Group theory*, chapter 10, Addison-Wesley, Reading, Mass. 1962.

[19] S. Kovacs, *A perturbative re-analysis of $N = 4$ supersymmetric Yang-Mills theory*, hep-th/9902047.

[20] D.Z. Freedman, K. Johnson, R. Muñoz-Tapia and X. Vilasis-Cardona, *A cutoff procedure and counterterms for differential renormalization*, *Nucl. Phys.* **B 395** (1993) 454 [hep-th/9206028].

[21] M.F. Sohnius, *Introducing supersymmetry*, *Phys. Rep.* **128** (1985) 39.

[22] G. Mack, A. Salam, *Finite component field representations of the conformal group*, *Ann. Phys. (NY)* **53** (1969) 174.

[23] O. Aharony, S.S. Gubser, J. Maldacena, H. Ooguri and Y. Oz, *Large-N field theories, string theory and gravity*, *Phys. Rep.* **323** (2000) 183 [hep-th/9905111].

[24] E. D'Hoker and A.V. Ryzhov, *Three-point functions of quarter BPS operators in $N = 4$ SYM*, hep-th/0109065.

JHEP11(2001)046